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BIAXIAL ISOTROPIC STOCHASTIC VISCO-ELASTIC CREEP

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Abstract—This is second part of a paper dealing with the formulation of constitutive equations for statistically isotropic multi-axial visco-elastic stochastic creep in terms of a second moment white noise field model. Herein the biaxial linearized model is studied. After an extensive retrospective introduction recapitulating the basic concepts, the paper presents the solution to the spatial covariance structure of a stress field history which is homogeneous in the mean. Results for the corresponding strain field are also presented. It turns out to be necessary to let the spatial second moment white noise character of the strain tensor field history for given deterministic stress tensor history be approached through a sequence of genuine covariance functions corresponding to isotropic random fields. In the limit the variances of both stress and strain become infinite. Interesting asymptotic results show up in this connection.

The appendices give some useful mathematical results concerning Fourier transforms related to the Laplace operator and covariance functions of isotropic random fields.

RETROSPECTIVE INTRODUCTION

The problem presentation in this introduction is related to papers on concrete creep. However, the considered stochastic model is of a general type that may be applicable to several materials.

Uniaxial creep

Modelling of concrete creep as a stochastic process seems to appear for the first time in a paper by Benjamin *et al.* [2]. The basic ideas of this pioneering paper was continued several years later for application on a more restricted problem by Cinlar *et al.* [3]. Their considerations solely deal with basic creep, i.e. creep that is not accompanied by moisture exchange, of a macroscopically homogeneous cylinder, and only creep under time-constant homogeneous uniaxial stress and temperature is considered.

Cinlar *et al.* argue that the creep function in the low stress domain is a stochastic process with nonnegative independent increments with respect to stress level and time. The increments with respect to the stress parameter are assumed to be stationary. Furthermore the process of displacements of cross sections along the cylinder is assumed to be a process with independent and stationary increments. These assumptions are also the basis for two recent papers by the writer [5, 6] and the present paper. The assumptions are of a general nature as compared to a further assumption introduced by Cinlar *et al.* They assume that basic creep is a local gamma process in order to be able to define an explicit distribution family for the stochastic creep function process. However, an assumption of this type is not needed if interest is focused solely on the second moment properties of the process.

For the purpose of first and second moment calculation the increment of any process $X(t)$ with uncorrelated increments (independent increments $\Rightarrow$ uncorrelated increments) may be written formally as an integral

$$X(t) - X(s) = \int_s^t W(\tau)\, d\tau \qquad (1)$$

in which $W(\tau)$ is a mathematical idealization called a "second moment white noise process" (or "wide sense white noise process"). Within the concept of mean square convergence ([7], p. 277) the integral makes sense if there is given a mean value function $m(\tau)$, i.e.

$$E[W(\tau)] = m(\tau) \tag{2}$$

and a nonnegative function $c(\tau)$ such that

$$\mathrm{Cov}[W(\tau), W(\theta)] = c(\tau)\,\delta(\theta - \tau) \tag{3}$$

in which $\delta(\cdot)$ is the Dirac delta function. The value $c(\tau)$ is called the intensity of the white noise to time τ. The usual rules of second moment calculus for processes then apply to give the mean increment

$$E[X(t) - X(s)] = \int_s^t m(\tau)\,\mathrm{d}\tau \tag{4}$$

and the covariance

$$\begin{aligned}\mathrm{Cov}[X(t) - X(s), X(v) - X(u)] &= \int_s^t \int_u^v \mathrm{Cov}[W(\tau), W(\theta)]\,\mathrm{d}\tau\,\mathrm{d}\theta \\ &= \int_s^t c(\tau) \int_u^v \delta(\theta - \tau)\,\mathrm{d}\theta\,\mathrm{d}\tau = \int_s^t c(\tau)\,1_{]u,v[}(\tau)\,\mathrm{d}\tau \\ &= \int_{\max\{s,u\}}^{\min\{t,v\}} c(\tau)\,\mathrm{d}\tau = \mathrm{Var}[X(\min\{t, v\}) - X(\max\{s, u\})]\end{aligned} \tag{5}$$

for $\max\{s, u\} \le \min\{t, v\}$, and 0 otherwise. The function $1_{]u,v[}(\tau)$ is the indicator function for the interval $]u, v[$, i.e. it is 1 if $\tau \in]u, v[$ and 0 otherwise.

On this level of modelling the sample function behavior of $X(t)$ is irrelevant except that a requirement that $X(t)$ has nonnegative increments implies that the mean value function $m(\tau)$ must be nonnegative.

Remark 1. White noise is most often defined to be Gaussian in the sense that $X(t)$ is a Gaussian process with independent increments. However, this is not a process with nonnegative increments and it is therefore not applicable as a model for stochastic creep. For this purpose the white noise must be nonnegative. It can be constructed by limit passage of a sequence of stationary lognormal processes obtained as exponentials of a sequence of stationary Gaussian processes that approaches Gaussian white noise. The resulting process with independent increments defined by use of eqn (1) is distributionally very complicated. Another much simpler type of nonnegative white noise is obtained by letting the increments of $X(t)$ be gamma distributed. This is used by Cinlar *et al.* [3]. To emphasize that distributional assumptions and detailed sample function behavior is of no concern for the pure second moment calculus we use the terminology "second moment white noise process" noting that this concept encompasses nonnegative white noise processes.

The advantage of expressing a process with uncorrelated increments as an integral of a white noise process becomes more obvious when generalizing to several parameters. For the uniaxial creep case the writer [5] has modelled the strain $\epsilon(\mathbf{r}, t)$ at the point $\mathbf{r} = (x, y, z)$ to time t by the formal integral

$$\epsilon(\mathbf{r}, t) = \int_{\tau=0}^{t} \int_{\sigma(\mathbf{r},\tau)}^{\sigma(\mathbf{r},\tau)+\mathrm{d}\sigma(\mathbf{r},\tau)} S(\mathbf{r}, t, \tau, u)\,\mathrm{d}u \tag{6}$$

in which $S(\mathbf{r}, t, \tau, u)$ is a second moment white noise process with respect to the parameters $\mathbf{r}$, τ and u. The parameter τ is the time of applying the stress increment $\mathrm{d}\sigma(\mathbf{r}, \tau)$ at the place $\mathbf{r}$, and u is the stress level parameter. Equations (2) and (3) simply generalize to

$$E[S(\mathbf{r}, t, \tau, u)] = K(t, \tau) \tag{7}$$

$$\mathrm{Cov}[S(\mathbf{r}_1, t_1, \tau_1, u_1), S(\mathbf{r}_2, t_2, \tau_2, u_2)] = c(t_1, t_2, \tau_1)\,\delta(\tau_2 - \tau_1)\,\delta(u_2 - u_1)\,\delta(\mathbf{r}_2 - \mathbf{r}_1) \tag{8}$$

in which $K(t, \tau)$ and $c(t_1, t_2, \tau)$ are nonnegative functions, and $\delta(\mathbf{r}_2 - \mathbf{r}_1) = \delta(x_2 - x_1)\,\delta(y_2 - y_1)\,\delta(z_2 - z_1)$.

The general assumptions of Cinlar *et al.* concerning the stochastic strain variation with respect to the length parameter and the stress level (i.e. their nondistributional assumptions) are in generalized form the sole basis for eqn (6). The generalization is simply to a beam-column type of specimen subjected to an arbitrary local stress history allowing for considering any nonhomogeneous uniaxial stress field history in the specimen. Using the same principles of calculation as demonstrated by eqns (4) and (5) gives [5]

$$E[\epsilon(\mathbf{r}, t)] = \int_{\tau=0}^{t} K(t, \tau)\, \mathrm{d}\sigma(\mathbf{r}, \tau) \tag{9}$$

$$\mathrm{Cov}[\epsilon(\mathbf{r}_1, s), \epsilon(\mathbf{r}_2, t)] = \delta(\mathbf{r}_2 - \mathbf{r}_1) \int_{\tau=0}^{s} c(s, t, \tau)\, |\mathrm{d}\sigma(\mathbf{r}_1, \tau)| \tag{10}$$

for $s \le t$. Note that the variance is nondecreasing since it is dependent on the stress increment solely through its absolute value. The mean decreases when the stress increment is negative, of course.

Equation (6) is in its formulation not restricted to the assumption that the strain process is a process with uncorrelated increments with respect to time t. If we impose this restriction assumed by Cinlar *et al.*, we may write

$$S(\mathbf{r}, t, \tau, u) = \int_{s=\tau}^{t} W(\mathbf{r}, s, \tau, u)\, \mathrm{d}s \tag{11}$$

in which $W(\mathbf{r}, t, \tau, u)$ is second moment white noise with respect to all parameters. Its mean and intensity are given by nonnegative functions $k(t, \tau)$ and $c_W(t, \tau)$ respectively. The implication is that

$$K(t, \tau) = \int_{\tau}^{t} k(s, \tau)\, \mathrm{d}s \tag{12}$$

and that the function $c(s, t, \tau)$ is only a function of τ and the smallest of s and t, i.e. $c(s, t, \tau) = c_S(\min\{s, t\}, \tau)$ where

$$c_S(t, \tau) = \int_{\tau}^{t} c_W(s, \tau)\, \mathrm{d}s. \tag{13}$$

The question of restricting $S(\mathbf{r}, t, \tau, u)$ to be of the form as in eqn (11) is somewhat controversial. The literature shows no general agreement on this. Benjamin *et al.* [2] assume that there are at least two independent stochastic creep components, viscous creep and delayed elasticity. This assumption is in line with the widely used separation into components in deterministic modelling of concrete creep. The separation is questioned by Bazant [1] and it is not made in the paper by Cinlar, Bazant and Osman [3]. Only the dominating viscous part is by Benjamin *et al.* modelled in terms of a process with independent increments. The delayed elasticity part is modelled as a Markov birth (or death) process which, however, is not a process with independent (or uncorrelated) increments. On the other hand, the delayed elasticity model of Benjamin *et al.* contributes to the total variance in a transient way, i.e. it vanishes asymptotically after a certain growth time. Thus it may be neglected without causing any essential error in the long term variance.

Example 1. It may be illustrative to go through an elementary alternative derivation of the model of Cinlar *et al.* Let g be a nonnegative function such that

$$G(\delta) = \int_{\delta}^{\infty} g(y)\, \mathrm{d}y < \infty \quad \text{for} \quad \delta > 0 \tag{14}$$

and $G(0) = \infty$. Assume that micro creep events take place within a unit length of the cylinder and within the time interval $[s, t]$ as a Poisson process with intensity $a\, G(\delta b)\, d\sigma$, where a, b, and δ are positive constants, while $d\sigma$ is the stress increment applied at time τ. In order to simplify writing we put $d\sigma$ to the stress unit in the following calculations. Assume that the ith creep event after time $s \geq \tau$ causes a random strain Y_i with density function

$$f_Y(y) = \frac{bg(by)}{G(\delta b)}, \qquad y \geq \delta. \tag{15}$$

Let N be the number of creep events occurring in the time interval $[s, t]$. Then the strain increment is

$$\epsilon(t) - \epsilon(s) = Y_1 + Y_2 + \cdots + Y_N \tag{16}$$

with conditional mean $E[\epsilon(t) - \epsilon(s) \mid N] = N\, E[Y]$ and variance $\text{Var}[\epsilon(t) - \epsilon(s) \mid N] = N\, \text{Var}[Y]$. Thus by the total representation theorem [7, p. 56]

$$E[\epsilon(t) - \epsilon(s)] = E[N]\, E[Y] = a\, G(\delta b)\, E[Y]\, (t - s) \tag{17}$$

$$\begin{aligned} \text{Var}[\epsilon(t) - \epsilon(s)] &= \text{Var}[N]\, E[Y]^2 + E[N]\, \text{Var}[Y] \\ &= a\, G(\delta b)\, E[Y^2]\, (t - s). \end{aligned} \tag{18}$$

It follows from eqn (15) that

$$G(\delta b)\, E[Y] = \frac{1}{b} \int_{\delta b}^{\infty} x\, g(x)\, dx \propto \frac{A}{b} \tag{19}$$

$$G(\delta b)\, E[Y^2] = \frac{1}{b^2} \int_{\delta b}^{\infty} x^2\, g(x)\, dx \propto \frac{B}{b^2} \tag{20}$$

asymptotically for $\delta \rightarrow 0$ provided

$$A = \int_0^{\infty} x\, g(x)\, dx < 0, \quad B = \int_0^{\infty} x^2\, g(x)\, dx < \infty. \tag{21}$$

Under this assumption eqns (17) and (18) give

$$E[\epsilon(t) - \epsilon(s)] \propto A\, \frac{a}{b}(t - s) \tag{22}$$

$$\text{Var}[\epsilon(t) - \epsilon(s)] \propto B\, \frac{a}{b^2}(t - s) \tag{23}$$

and we get the coefficient of variation

$$V_{\epsilon(t) - \epsilon(s)} \propto \frac{\sqrt{B}}{A}\, \frac{1}{\sqrt{a(t - s)}} \tag{24}$$

asymptotically for $\delta \rightarrow 0$. We see that the uncertainty of the creep strain by this model shows up macroscopically even though both the mean $E[Y]$ and the variance $\text{Var}[Y]$ of the single creep event according to eqns (19) and (20) approach zero for $\delta \rightarrow 0$ because $G(0) = \infty$. This vanishing effect of the single creep event is counteracted by the large intensity of the Poisson process approaching infinity as $\delta \rightarrow 0$.

In the model of Cinlar *et al.* the function g is explicitely defined by

$$g(y) = \frac{e^{-y}}{y}, \qquad y > 0 \tag{25}$$

in which case $A = B = 1$. The Laplace transform $\mathcal{L}(\lambda)$ of $f_Y(y)$ becomes

$$\mathcal{L}(\lambda) = E[e^{-\lambda Y}] = \frac{1}{G(\delta b)} \int_{\delta}^{\infty} \frac{1}{y}\, e^{-(b+\lambda)y}\, dy = \frac{G[\delta(b + \lambda)]}{G(\delta b)} \tag{26}$$

such that the Laplace transform corresponding to $Y_1 + \cdots Y_N$ with $h = t - s$ becomes

$$\begin{aligned} E[e^{-\lambda(Y_1 + \cdots + Y_N)}] &= \sum_{n=0}^{\infty} \left(\frac{G[\delta(b + \lambda)]}{G(\delta b)}\right)^n \frac{[ahG(\delta b)]^n}{n!}\, e^{-ahG(\delta b)} \\ &= e^{-ahG(\delta b)} \sum_{n=0}^{\infty} \frac{1}{n!} (ahG[\delta(b + \lambda)])^n \\ &= \exp\left[-ah\{G(\delta b) - G[\delta(b + \lambda)]\}\right] \end{aligned} \tag{27}$$

in which

$$G(\delta b) - G[\delta(b+\lambda)] = \int_{\delta b}^{\delta(b+\lambda)} \frac{e^{-y}}{y}\,dy \to \log\frac{b+\lambda}{b} \tag{28}$$

such that

$$E[e^{-\lambda(Y_1+\cdots+Y_N)}] \to \left(\frac{b}{b+\lambda}\right)^{ah} \tag{29}$$

for $\delta \to 0$. The limit is the Laplace transform of the gamma distribution with density

$$\frac{b}{\Gamma(ah)}(by)^{ah-1}e^{-by}, \qquad y > 0. \tag{30}$$

From the continuity theorem of Laplace transforms ([8], p. 408) it thus follows that the strain increment $\epsilon(t) - \epsilon(s)$ for the limiting model obtained for $\delta \to 0$ is gamma distributed with scale parameter b and shape parameter ah. This, in fact, is the local gamma process model of Cinlar *et al.* corresponding to the strain increment over the time increment h at time s. Letting a and b be functions of s and τ lead to a definition of the entire creep function process for a unit stress increment applied at time τ. For a nonnegative stress increment $d\sigma$ different from the stress unit we only need to multiply a by $d\sigma$ in eqn (30).

If we adopt the white noise model of eqns (6) and (11), the mean and variance of the average strain

$$\epsilon_{\mathcal{B}}(t) = \frac{1}{|\mathcal{B}|}\int_{\mathbf{r}\in\mathcal{B}} \epsilon(\mathbf{r}, t) \tag{31}$$

over a subbody $\mathcal{B}$ of the specimen (notation: $|\mathcal{B}|$ = volume of $\mathcal{B}$) by substitution of eqns (12) and (13) into eqns (9) and (10) respectively become

$$E[\epsilon_{\mathcal{B}}(t)] = \int_{\tau=0}^{t}\left(\int_{s=\tau}^{t} k(s,\tau)\,ds\right)d\sigma(\tau) \tag{32}$$

$$\mathrm{Var}[\epsilon_{\mathcal{B}}(t)] = \frac{1}{|\mathcal{B}|}\int_{\tau=0}^{t}\left(\int_{s=\tau}^{t} c_W(s,\tau)\,ds\right)|d\sigma(\tau)| \tag{33}$$

assuming by writing $d\sigma(\tau) = d\sigma(\mathbf{r}, \tau)$ that the stress field is homogeneous within the body $\mathcal{B}$.

Obviously we may interpret the mean and variance in eqns (22) and (23) as corresponding to the average strain increment over a unit volume with an imposed homogeneous uniaxial unit stress field increment. By comparison of eqns (22) and (23) with eqns (32) and (33) we may therefore choose the functions $k(s, \tau)$ and $c_W(s, \tau)$ such that the constitutive equation (6) with eqn (11) substituted is consistent up to second moment results with the model of Cinlar *et al.* We get

$$k(s,\tau) = \frac{a(s,\tau)}{b(s,\tau)} \tag{34}$$

$$c_W(s,\tau) = \frac{a(s,\tau)}{b(s,\tau)^2}. \tag{35}$$

Example 2. We may even assign distributional properties to the white noise strain process $W(\mathbf{r}, s, \tau, u)$ that are consistent with the distributional assumption of Cinlar *et al.* The incremental contribution to the average strain $\epsilon_{\mathcal{B}}(t)$ for an increment in (s, τ, u) from the point $(s_1, \tau_1, \sigma(\tau_1))$ to the point $(s_2, \tau_2, \sigma(\tau_2))$ $(s_1 \le s_2, \tau_1 \le \tau_2, \tau_1 \le s_1, \tau_2 \le s_2$, Fig. 1) is

$$\frac{1}{|\mathcal{B}|}\int_{\mathbf{r}\in\mathcal{B}}\int_{\tau=\tau_1}^{\tau_2}\int_{u=\sigma(\mathbf{r},\tau)}^{\sigma(\mathbf{r},\tau)+d\sigma(\mathbf{r},\tau)}\int_{s=s_1}^{s_2} W(\mathbf{r}, s, \tau, u)\,ds\,du. \tag{36}$$

To this increment we may for $y\Delta\sigma > 0$ assign the probability density

$$\frac{b|\mathcal{B}|}{\Gamma(ah|\mathcal{B}||\Delta\sigma|)}\left(b|\mathcal{B}|\frac{\Delta\sigma}{|\Delta\sigma|}y\right)^{ah|\mathcal{B}||\Delta\sigma|-1}\exp\left[-b|\mathcal{B}|\frac{\Delta\sigma}{|\Delta\sigma|}y\right] \tag{37}$$

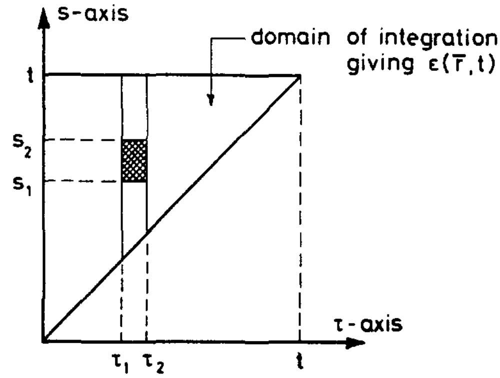

Fig. 1

and zero otherwise. Herein $h = t_2 - t_1$ is the time increment while $\Delta\sigma$ is the stress increment over the increment $\tau_2 - \tau_1$ in the time of loading τ. The density in eqn (37) is assumed to be the density of the increment in eqn (36) asymptotically for $h \to 0$, $\tau_2 - \tau_1 \to 0$ and $|\mathscr{B}| \to 0$ such that the diameter of $\mathscr{B}$ approaches zero. The distribution of $\epsilon_{\mathscr{B}}(t)$ is finally assigned as the convolution of the densities of eqn (37) over all increments making up $\epsilon_{\mathscr{B}}(t)$. It is not an easy task to calculate this convolution for a general stress history unless b is a constant and $\Delta\sigma/|\Delta\sigma| = 1$ (or -1) for all τ (setting $\Delta\sigma/|\Delta\sigma| = 1$ (or -1) whenever $\Delta\sigma = 0$). In that case $\epsilon_{\mathscr{B}}(t)$ (or $-\epsilon_{\mathscr{B}}(t)$) has a gamma distribution with scale parameter $b\,|\mathscr{B}|$ and shape parameter

$$\int_{r\in\mathscr{B}} \int_{\tau=0}^{t} \int_{s=\tau}^{t} a(s, \tau)\, ds\, |d\sigma(\mathbf{r}, \tau)| \tag{38}$$

in which $\mathscr{B}$ is not restricted to be small.

Triaxial creep

The scalar constitutive equation, eqn (6), may be generalized to the tensor equation

$$\epsilon_{ij}(\mathbf{r}, t) = \int_{\tau=0}^{t} \int_{\bar{u}=\bar{\sigma}(\mathbf{r},\tau)}^{\bar{\sigma}(\mathbf{r},\tau)+d\bar{\sigma}(\mathbf{r},\tau)} S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})\, du_{rs} \tag{39}$$

giving the creep strain tensor $\epsilon_{ij}(\mathbf{r}, t)$ as function of an imposed stress tensor history $\bar{\sigma}(\mathbf{r}, \tau)$, $0 \le \tau \le t$. For fixed t the integrand $S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})$ is a second moment white noise random tensor process with parameter set $\mathscr{B} \times R_0 \times$ {space of stress tensors $\bar{u}$}. The writer [6] has explored the first and second moment properties of $S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})$ assuming statistical isotropy and a simplifying form-invariance property of the covariance tensor. Isotropy requires that the functional dependency of the stress tensor increment $d\bar{u} = \bar{u}_2 - \bar{u}_1$ is solely through the invariants of $d\bar{u}$. The simplifying assumption is that among the invariants the covariance tensor of S_{ijrs} is only a function of the norm $\| d\bar{u} \| = \sqrt{du_{ij}du_{ij}}$.

For the mean creep tensor the result is of the form as the usual constitutive equation for isotropic linear visco-elasticity:

$$\begin{aligned} E[\epsilon_{ij}(\mathbf{r}, t)] &= \int_{\tau=0}^{t} K_{ijrs}(t, \tau)\, d\sigma_{rs}(\mathbf{r},\tau) \\ &= \int_{\tau=0}^{t} (1 + \nu(t, \tau))\, C(t, \tau)\, d\sigma_{ij}(\mathbf{r}, \tau) \\ &\quad - \delta_{ij} \int_{\tau=0}^{t} \nu(t, \tau)\, C(t, \tau)\, d\sigma_{ss}(\mathbf{r}, \tau) \end{aligned} \tag{40}$$

in which δ_{ij} is Kronecker's delta, $C(t, \tau)$ is the creep function while $\nu(t, \tau)$ is the Poisson ratio function.

The covariance gets the form

$$\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \epsilon_{kl}(\mathbf{r}_2, t_2)] = \delta(\mathbf{r}_2 - \mathbf{r}_1) \int_0^{t_1} d_{ijklrspq}(t_1, t_2, \tau) \frac{d\sigma_{rs}(\mathbf{r}_1, \tau)\, d\sigma_{pq}(\mathbf{r}_1, \tau)}{\| d\bar{\sigma}(\mathbf{r}_1, \tau) \|} \tag{41}$$

in which the isotropy of the tensor $d_{ijklrspq}$ implies that the integrand in its most general form is given by 8 scalar functions $b_1, \ldots, b_8$ of t_1, t_2, τ:

$$d_{ijklrspq}\frac{d\sigma_{rs}\,d\sigma_{pq}}{\|d\bar{\sigma}\|} = [b_1\,d\sigma_{ij}\,d\sigma_{kl} + b_2\,(d\sigma_{ik}\,d\sigma_{jl} + d\sigma_{kj}\,d\sigma_{il}) + b_3\,(\delta_{ij}\,d\sigma_{kl} + \delta_{kk}\,d\sigma_{ij})\,d\sigma_{ss} + b_4\,(\delta_{jl}\,d\sigma_{ik} + \delta_{jk}\,d\sigma_{il} + \delta_{il}\,d\sigma_{jk} + \delta_{ik}\,d\sigma_{jl})\,d\sigma_{ss} + \delta_{ij}\delta_{kl}\,(b_5\,(d\sigma_{ss})^2 + b_6\,d\sigma_{rs}\,d\sigma_{rs}) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})\,(b_7\,(d\sigma_{ss})^2 + b_8\,d\sigma_{rs}\,d\sigma_{rs})]\frac{1}{\|d\bar{\sigma}\|}. \tag{42}$$

If the tensor version of eqn (11) is assumed to be valid, the implication is that the functions $b_1, \ldots, b_8$ become solely functions of $\min\{t_1, t_2\}$ and τ.

Example 3. Let a principal stress tensor increment of the form

$$d\sigma_{11} = \alpha\,d\sigma,\quad d\sigma_{22} = \beta\,d\sigma,\quad d\sigma_{33} = \gamma\,d\sigma,\quad d\sigma > 0,\qquad \alpha^2 + \beta^2 + \gamma^2 = 1 \tag{43}$$

be applied to time τ. Let $J(t, \tau)$ be a scalar nonnegative process in $t \geq \tau$ with mean $C(t, \tau)\,d\sigma$ and variance $w(t, \tau)\,d\sigma$. Assume that

$$\begin{aligned}\epsilon_{11} &= (\alpha - \beta\nu - \gamma\nu)\,J\\ \epsilon_{22} &= (-\alpha\nu + \beta - \gamma\nu)\,J\\ \epsilon_{33} &= (-\alpha\nu - \beta\nu + \gamma)\,J\end{aligned} \tag{44}$$

in which ν is some nonnegative deterministic function of t and τ. By comparison with eqn (40) it is seen that ν is the Poisson ratio function $\nu(t, \tau)$ while $C(t, \tau)$ is the creep function.

In order to have consistency between eqns (41), (42) and the variance

$$\mathrm{Var}[\epsilon_{11}] = (\alpha^2 + \beta^2\nu^2 + \gamma^2\nu^2)\,w\,d\sigma \tag{45}$$

for any normalized (α, β, γ) the functions $b_1, \ldots, b_8$ must be defined by

$$\begin{aligned}&b_1 = (1 + \nu)^2 w,\quad b_3 = -\nu(1 + \nu)w,\quad b_5 = \nu^2 w\\ &b_2 = b_4 = b_6 = b_7 = b_8 = 0.\end{aligned} \tag{46}$$

In Ref. [6] the process J is assumed to be proportional to a nonhomogeneous Poisson process of intensity $\mu(t, \tau)\,d\sigma$ with proportionality factor $a(t, \tau)$ (not the same $a(t, \tau)$ as in Examples 1 and 2). In that case the variance function w is $w = a^2\mu = aC$. However, this result is more general than the Poisson process assumption indicates because we may define the function a by

$$a(t, \tau) = \frac{w(t, \tau)}{C(t, \tau)} \tag{47}$$

for the function w corresponding to any nonnegative scalar process J of finite variance. In place of the terminology "Poisson process viscous creep" used in Ref. [6] we will therefore simply use the more general terminology "scalar process creep" in case the functions $b_1, \ldots, b_8$ are defined by eqn (46).

The strain tensor $\epsilon_{ij}(\mathbf{r}, t)$ must satisfy the usual local compatibility conditions and the stress tensor increment must satisfy the local equilibrium conditions. Since the constitutive tensor equation, eqn (39), is stochastic, it follows that not only the strain history is stochastic but also the stress history has this property due to the compatibility conditions. Thus the stress field cannot be imposed as a free variable. Only stresses on the surface of the body and/or displacements of the surface can be specified either in terms of given random fields or given deterministic functions of position on the surface. This fact makes eqn (39) nonlinear in the stochastic processes $\bar{\sigma}$ and S_{ijrs} such that it becomes virtually impossible to solve any boundary value problem. Therefore it is necessary to substitute a simpler constitutive equation for eqn (39) obtained from eqn (39) by linearization with respect to $\bar{\sigma}$, $\bar{\sigma} + d\bar{\sigma}$, and S_{ijrs}. This linearized constitutive tensor equation reads [6]

$$\epsilon_{ij}(\mathbf{r}, t) = \int_{\tau=0}^{t} \int_{E[\bar{\sigma}(\mathbf{r},\tau)]}^{E[\bar{\sigma}(\mathbf{r},\tau)] + dE[\bar{\sigma}(\mathbf{r},\tau)]} S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})\, du_{rs} + \int_{\tau=0}^{t} K_{ijrs}(t, \tau)\,(d\sigma_{rs}(\mathbf{r}, \tau) - dE[\sigma_{rs}(\mathbf{r}, \tau)]) \tag{48}$$

in which $K_{ijrs}(t, \tau) = E[S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})]$.

Biaxial creep

This paper considers the simplest possible biaxial stress problem. It deals with the determination of the covariance structure of a stochastically homogeneous stress field and the corresponding strain field. It should be emphasized that strictly there exists no biaxial state of stress within the three-dimensional model except in the mean. The compatibility constraints will cause stresses to develop with components in all three directions of the space. In order to be exact when speaking of a biaxial state of stress it should, in fact, be related to a two-dimensional model. A two-dimensional model is, however, given directly from the three-dimensional model by restricting the index set to $\{1, 2\}$.

Some conclusions. The experimental challenge

The detailed biaxial stress analysis to follow reveals some interesting features of the stochastic creep model defined by the linearized constitutive equation, eqn (48), under due consideration of the local compatibility requirements. Whether these features reflect real creep behavior of a material like concrete or any other material is an open question, of course, which may be a challenge to experimentalists in mechanics.

The first conclusion is that the assumption that the tensor process $S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})$ is white noise with respect to the two-dimensional space variable $\mathbf{r}$ implies that both stresses and strains of a stochastically homogeneous stress field get infinite variance without the fields being neither white noise in the plane or along curves in the plane.

In order to get finite variance solutions it is necessary to substitute an ordinary covariance function in place of the delta function $\delta(\mathbf{r}_2 - \mathbf{r}_1)$ in eqn (41). It may be taken as the generic member $\rho_\sigma(\mathbf{r}_2 - \mathbf{r}_1)$ of the sequence of functions by which the Dirac delta function is defined as limit. Specifically we have

$$\rho_\sigma(\mathbf{r}) \rightarrow \delta(\mathbf{r}) \quad \text{for} \quad \sigma \rightarrow 0 \tag{49}$$

in which ρ_σ, $\sigma \in R_+$, is a family of functions that are admissible covariance functions for homogeneous and isotropic random fields, and which have the property

$$\int_{R^2} \rho_\sigma(x_1^2 + x_2^2)\, f(x_1, x_2)\, dx_1\, dx_2 \rightarrow f(0, 0) \tag{50}$$

for $\sigma \rightarrow 0$ for any function f continuous at (0, 0) for which the integral exists. In particular, the function $f(x_1, x_2) \equiv 1$ is assumed to belong to this class of functions. Specific examples of such families of covariance functions are given in eqns (66) and (67). The isotropy ensures that $\rho_\sigma(\mathbf{r})$ is of the form

$$\rho_\sigma(\mathbf{r}) = \frac{1}{\sigma^2}\, \rho\left(\frac{x_1^2 + x_2^2}{\sigma^2}\right) \tag{51}$$

where $\rho(x)$, $x \geq 0$, is a function with $\rho(0) < \infty$, $x\,\rho(x) \rightarrow 0$ for $x \rightarrow \infty$, and $\pi \int_0^\infty \rho(x)\, dx = 1$. After the substitution of $\rho_\sigma(\mathbf{r}_2 - \mathbf{r}_1)$ for $\delta(\mathbf{r}_2 - \mathbf{r}_1)$ in eqn (41) the equation still represents an admissible correlation structure of the strain field ([7], p. 359).

This step of changing the delta function factor in eqn (41) to a generic member of its defining sequence is in line with the approximation philosophy behind the use of the concept of second moment white noise field. This concept is introduced as a mathematically simplifying approximation to a random field with a "small" correlation length scale ([5], pp. 23–24). Thus using $\rho_\sigma(\mathbf{r}_2 - \mathbf{r}_1)$ in place of $\delta(\mathbf{r}_2 - \mathbf{r}_1)$ with "small" σ corresponds to a step back toward the original correlation structure of the random field.

For $\rho_\sigma(\mathbf{r})$ given on the general form of eqn (51) a solution is obtained for the covariance structure of the stress tensor field and the strain tensor field. It is of particular interest to study the behavior of the solutions as $\sigma \to 0$, of course. It turns out that the variance of the average normal stress or the average shear stress on a linear cut of length L is asymptotically proportional to $\log(L/\sigma)/L^2$ for large L/σ. After division of the variance by $\log(1/\sigma) = -\log\sigma$ we therefore get a finite limit proportional to $1/L^2$ for $\sigma \to 0$. This behavior (and also the size of the proportionality factor) is independent of the particular function ρ used in the limit operation.

The consequence of this observation is as follows. Let the covariance structure of the second moment white noise tensor process S_{ijrs} of the constitutive equation be changed as follows. Replace the delta function $\delta(\mathbf{r}_2 - \mathbf{r}_1)$ in eqn (41) as being the generalized function obtained from $\rho_\sigma(\mathbf{r}_2 - \mathbf{r}_1)$ in the limit $\sigma \to 0$ by a type of generalized function obtained by passing to the limit with $\rho_\sigma(\mathbf{r}_2 - \mathbf{r}_1)/(-\log\sigma)$ for $\sigma \to 0$. In this way we obtain a formal process which may be called a *logarithmically weakened* second moment white noise process. For this model the average strain across any finite part of nonzero volume of the body caused by an imposed deterministic stress history will be deterministic in the sense that its variance is zero. However, the compatibility conditions make it impossible to impose a deterministic stress history. A random stress field will develop giving an average stress across a linear cut of length L with finite standard deviation proportional to $1/L$. Furthermore, it turns out that the stress field will cause strains that in general also after integration along a path will be of nonzero variance. The perspective is quite interesting. In Ref. [6] it was shown that the covariance properties of the average strain tensor across a body subjected solely to given external forces for the linearized constitutive model are determined by the mean stress field history. If the logarithmically weakened white noise model is adopted, the consequence therefore is, e.g., that the elongation and curvature processes of the beam of Example 3 in Ref. [6] become deterministic. This shows that it is not given that the usual standard measurements of deformations of external statically determinate test pieces are well suited to disprove the existence of phenomena that may be modelled as random visco-elasticity in the sense defined herein.

Another interesting conclusion concerns the displacements relative to the origin. The variance of the radial relative displacement off the mean stress principal axes of a point in distance L from the origin is in general asymptotically proportional to L/σ for large L/σ. In some cases as for example for scalar process creep, see Example 3, the variance increases at less order of magnitude than L/σ for increasing L/σ. When the radial direction is coincident with any one of the mean stress principal axes, the variance of the relative displacement is asymptotically proportional to $\log(L/\sigma)$ for large L/σ. For the logarithmically weakened second moment white noise model the consequence of this peculiar behavior is that the variance of the relative displacements in any of the two directions of the mean stress principal axes for $\sigma \to 0$ will approach a finite and constant value while it may diverge toward infinity for all other directions.

HOMOGENEOUS RANDOM BIAXIAL STRESS FIELD

For a homogeneous random stress field the mean stress tensor increment $dE[\sigma_{rs}(\mathbf{r}, \tau)]$ is independent of $\mathbf{r}$ and all covariances are dependent on $\mathbf{r}_1, \mathbf{r}_2$ only in terms of the difference $\mathbf{r}_2 - \mathbf{r}_1$. Since the stress field is biaxial, the equilibrium equations are automatically satisfied by defining the random stress increments in terms of a random increment of an Airy stress function $\Phi(\mathbf{r}, t)$. We have

$$\begin{aligned} d\sigma_{11}(\mathbf{r}, \tau) &= d\Phi_{,22}(\mathbf{r}, \tau) \\ d\sigma_{22}(\mathbf{r}, \tau) &= d\Phi_{,11}(\mathbf{r}, \tau) \\ d\sigma_{12}(\mathbf{r}, \tau) &= -d\Phi_{,12}(\mathbf{r}, \tau). \end{aligned} \tag{52}$$

There is only one compatibility equation

$$\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0 \tag{53}$$

which after substitution of ϵ_{ij} given by eqn (48) becomes

$$\int_{\tau=0}^{t} \int_{E[\bar{\sigma}(\mathbf{r},\tau)]}^{E[\bar{\sigma}(\mathbf{r},\tau)] + dE[\bar{\sigma}(\mathbf{r},\tau)]} (S_{11rs,22} + S_{22rs,11} - 2S_{12rs,12})(\mathbf{r}, t, \tau, \bar{u})\, du_{rs}$$
$$+ \int_{\tau=0}^{t} (K_{11rs}\, d\sigma_{rs,22} + K_{22rs}\, d\sigma_{rs,11} - 2K_{12rs}\, d\sigma_{rs,12})(\mathbf{r}, t, \tau) = 0. \tag{54}$$

By the notation $\bar{m}(\tau) = E[\bar{\sigma}(\mathbf{r}, \tau)]$ and by using eqns (40) and (52), eqn (54) may be rewritten as

$$\int_{\tau=0}^{t} C(t, \tau)\, (\Delta^2)_{\mathbf{r}}\, d\Phi(\mathbf{r}, \tau)$$
$$= -\int_{\tau=0}^{t} \int_{\bar{m}(\tau)}^{\bar{m}(\tau) + d\bar{m}(\tau)} (S_{11rs,22} + S_{22rs,11} - 2S_{12rs,12})(\mathbf{r}, t, \tau, \bar{u})\, du_{rs} \tag{55}$$

in which $(\Delta^2)_{\mathbf{r}}$ is the squared Laplace operator, i.e.

$$(\Delta^2)_{\mathbf{r}}\, \Phi = \Phi_{,1111} + 2\Phi_{,1122} + \Phi_{,2222}. \tag{56}$$

The covariance function of the left side of eqn (55) is

$$\mathrm{Cov}\left[\int_{\tau_1=0}^{t_1} C(t_1, \tau_1)(\Delta^2)_{\mathbf{r}_1}\, d\Phi(\mathbf{r}_1, \tau_1),\ \int_{\tau_2=0}^{t_2} C(t_2, \tau_2)(\Delta^2)_{\mathbf{r}_2}\, d\Phi(\mathbf{r}_2, \tau_2) \right]$$
$$= (\Delta^2)_{\mathbf{r}_1}(\Delta^2)_{\mathbf{r}_2} \int_{\tau_1=0}^{t_1} \int_{\tau_2=0}^{t_2} C(t_1, \tau_1)C(t_2, \tau_2)\, \mathrm{Cov}[d\Phi(\mathbf{r}_1, \tau_1), d\Phi(\mathbf{r}_2, \tau_2)]. \tag{57}$$

By the homogeneity assumption this double integral is a function of $\mathbf{r}_1$, $\mathbf{r}_2$ solely through the difference $\mathbf{r}_2 - \mathbf{r}_1$. Writing this function as $F(\mathbf{r}_2 - \mathbf{r}_1, t_1, t_2)$, i.e.

$$F(\mathbf{r}_2 - \mathbf{r}_1, t_1, t_2) = \int_{\tau_1=0}^{t_1} \int_{\tau_2=0}^{t_2} C(t_1, \tau_1)C(t_2, \tau_2)\, \mathrm{Cov}[d\Phi(\mathbf{r}_1, \tau_1), d\Phi(\mathbf{r}_2, \tau_2)] \tag{58}$$

it follows that the left side of eqn (55) has the covariance function

$$\Delta^4 F(\mathbf{x}, t_1, t_2) \tag{59}$$

in which $\mathbf{x} = \mathbf{r}_2 - \mathbf{r}_1$.

The right side of eqn (55) has a covariance function that follows by use of eqn (41) in which the integrand is independent of $\mathbf{r}_1$, $\mathbf{r}_2$ when $dm_{rs}(\tau)$ is substituted for $d\sigma_{rs}(\mathbf{r}_1, \tau)$, $dm_{rs}(\tau)$ being the components of $d\bar{m}(\tau)$. In the following the coordinate system of principal mean stress increments is kept time invariant. Writing the contracted tensor in the integrand of eqn (41) as d_{ijkl} and referring to the coordinate system of principal mean stresses, the covariance function becomes

$$\int_0^{t_1} [d_{1111}\,\delta(x_1)\,\delta^{(4)}(x_2) + (d_{1122} + d_{2211} + 4d_{1212})\,\delta^{(2)}(x_1)\,\delta^{(2)}(x_2) + d_{2222}\,\delta^{(4)}(x_1)\,\delta(x_2)]. \quad (60)$$

Thus by equating the expressions of eqns (59) and (60) we get the partial differential equation

$$\Delta^4 F(\mathbf{x}) = \sum_{p=0}^{2} a_{2p}\,\delta^{(2p)}(x_1)\,\delta^{(4-2p)}(x_2) \quad (61)$$

in which

$$a_0 = \int_0^{t_1} d_{1111}, \quad a_2 = 2\int_0^{t_1} (d_{1122} + 2d_{1212}), \quad a_4 = \int_0^{t_1} d_{2222} \quad (62)$$

are given functions of (t_1, t_2). In order to keep the notation short, t_1, t_2 are not shown explicitly in eqn (61) and in the following. Writing $F(x_1, x_2)$ symbolically in terms of its Fourier transform $\hat{F}(\omega_1, \omega_2)$, i.e.

$$F(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{F}(\omega_1, \omega_2)\,e^{i(x_1\omega_1 + x_2\omega_2)}\,d\omega_1\,d\omega_2 \quad (63)$$

and the Dirac delta function as

$$\delta(x_1)\delta(x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^2 e^{i(x_1\omega_1 + x_2\omega_2)}\,d\omega_1\,d\omega_2 \quad (64)$$

(remembering that these integrals are the Cauchy principal values) it is seen by formal differentiations behind the integral signs followed by substitution of the obtained derivatives into eqn (61) that

$$\hat{F}(\omega_1, \omega_2) = \left(\frac{1}{2\pi}\right)^2 \sum_{p=0}^{2} a_{2p}\,\frac{\omega_1^{2p}\omega_2^{4-2p}}{(\omega_1^2 + \omega_2^2)^4}. \quad (65)$$

It turns out that this is not a Fourier transform of any ordinary or generalized function $F(x_1, x_2)$, see Appendix. This means that there exists no finite second moment solution corresponding to a homogeneous random stress field. As mentioned in the introduction it is, however, possible to obtain a solution if the delta function factor $\delta(x_1)\delta(x_2)$ ($= \delta(\mathbf{r}_2 - \mathbf{r}_1)$) in eqn (41) is changed to a function from a sequence of functions that defines the delta function in the limit. For example, such a function is

$$\frac{1}{2\pi\sigma^2}\exp\left[-\frac{x_1^2 + x_2^2}{2\sigma^2}\right] \quad (66)$$

which we will call a Gaussian type covariance function. Another example is

$$\frac{1}{2\pi\sigma^2}\left[1 + \left(\frac{x_1}{\sigma}\right)^2 + \left(\frac{x_2}{\sigma}\right)^2\right]^{-3/2} \quad (67)$$

which we will call Cauchy type covariance function. Both functions of eqns (66) and (67) give a delta function of (x_1, x_2) in the limit $\sigma \to 0$ and both are admissible covariance functions for isotropic random fields ([7], pp. 358–360). Thus eqn (41) after such a change still represents an admissible correlation structure of the strain field ([7], p. 359).

After change of $\delta(\mathbf{r}_2 - \mathbf{r}_1)$ to $\rho_\sigma(\mathbf{r}_2 - \mathbf{r}_1)$ where ρ_σ is given by eqn (51) as an admissible covariance function of an isotropic random field, eqn (61) is changed into

$$\Delta^4 F(\mathbf{x}) = \sum_{p=0}^{2} \frac{a_{2p}}{\sigma^2} \frac{\partial^4}{\partial x_1^{2p} \partial x_2^{4-2p}} \rho\left(\frac{x_1^2 + x_2^2}{\sigma^2}\right) \tag{68}$$

while eqn (65) is replaced by

$$\hat{F}(\omega_1, \omega_2) = \sum_{p=0}^{2} a_{2p} \frac{\omega_1^{2p}\omega_2^{4-2p}}{(\omega_1^2 + \omega_2^2)^4} \hat{\rho}((\sigma\omega_1)^2 + (\sigma\omega_2)^2) \tag{69}$$

where $\hat{\rho}(\omega_1^2 + \omega_2^2)$ is the Fourier transform of $\rho(x_1^2 + x_2^2)$. In order to calculate covariances between any two stress tensor components it follows from eqns (52) and (58) that we need the inverse transform of

$$\omega_1^q \, \omega_2^{4-q} \, \hat{F}(\omega_1, \omega_2) \tag{70}$$

for $q = 0, 1, 2, 3, 4$. Writing the function corresponding to the Fourier transform

$$\hat{F}_{(\alpha,\beta)}(\omega_1, \omega_2) = \frac{\omega_1^{\alpha} \, \omega_2^{4\beta-\alpha}}{(\omega_1^2 + \omega_2^2)^{2\beta}} \hat{\rho}(\omega_1^2 + \omega_2^2) \tag{71}$$

as $F_{(\alpha,\beta)}(x_1, x_2)$, it follows from eqn (69) that

$$\frac{\partial^4 F(x_1, x_2)}{\partial x_1^q \partial x_2^{4-q}} = \frac{1}{\sigma^2} \sum_{p=0}^{2} a_{2p} \, F_{(2p+q,2)}\left(\frac{x_1}{\sigma}, \frac{x_2}{\sigma}\right). \tag{72}$$

The particular form of the right side of eqn (72) and the linearity of eqn (58) imply that

$$\mathrm{Cov}[\mathrm{d}\sigma_{ij}(\mathbf{r}_1, \tau_1),\ \mathrm{d}\sigma_{kl}(\mathbf{r}_2, \tau_2)] = (-1)^q \frac{\partial^4}{\partial x_1^q \partial x_2^{4-q}} \mathrm{Cov}[\mathrm{d}\Phi(\mathbf{r}_1, \tau_1),\ \mathrm{d}\Phi(\mathbf{r}_2, \tau_2)] \tag{73}$$

$$q = i + j + k + l - 4 \tag{74}$$

may be written as

$$(-1)^q \frac{1}{\sigma^2} \sum_{p=0}^{2} A_{2p}(\tau_1, \tau_2) \, F_{(2p+q,2)}\left(\frac{x_1}{\sigma}, \frac{x_2}{\sigma}\right) \tag{75}$$

in which $A_{2p}(\tau_1, \tau_2)$ is the solution to the integral equation

$$\int_{\tau_1=0}^{t_1} \int_{\tau_2=0}^{t_2} C(t_1, \tau_1) \, C(t_2, \tau_2) \, A_{2p}(\tau_1, \tau_2) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 = a_{2p}(t_1, t_2). \tag{76}$$

This equation may be solved by numerical standard technique approximating it by a set of linear equations. Having determined the functions $A_0(\tau_1, \tau_2)$, $A_2(\tau_1, \tau_2)$, $A_4(\tau_1, \tau_2)$ we finally get

$$\begin{aligned}&\mathrm{Cov}[\sigma_{ij}(\mathbf{r}_1, t_1), \sigma_{kl}(\mathbf{r}_2, t_2)] \\ &= (-1)^q \frac{1}{\sigma^2} \sum_{p=0}^{2} \left(\int_{\tau_1=0}^{t_1} \int_{\tau_2=0}^{t_2} A_{2p}(\tau_1, \tau_2) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \right) F_{(2p+q,2)}\left(\frac{x_1}{\sigma}, \frac{x_2}{\sigma}\right)\end{aligned} \tag{77}$$

with q given by eqn (74).

AVERAGE NORMAL STRESS AND SHEAR STRESS ON A LINEAR CUT

Let a linear cut be given by the parametric representation $(x_1, x_2) = (as, bs)$ where a, b are constants for which $a^2 + b^2 = 1$ and $s \in R$ is the parameter. Then the normal stress on the cut is given by $\sigma_{ij}\, n_i\, n_j$ where $n_1 = -b$, $n_2 = a$, while the shear stress is given by $\sigma_{ij}\, r_i\, t_j$ where $t_1 = a$, $t_2 = b$. Denoting by superscripts I and II the stress tensor at place and time I and place and time II along the cut and in mutual distance s we have from eqn (77):

$$\begin{aligned}\mathrm{Cov}[\sigma^I_{ij}\, n_i\, n_j, \sigma^{II}_{kl}\, n_k\, n_l] &= n_i\, n_j\, n_k\, n_l\, \mathrm{Cov}[\sigma^I_{ij}, \sigma^{II}_{kl}] \\ &= \frac{1}{\sigma^2} \sum_{p=0}^{2} \left(\int_{\tau_1=0}^{t_1} \int_{\tau_2=0}^{t_2} A_{2p}(\tau_1, \tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \right) \\ &\quad \times (-1)^{i+j+k+l} F_{(2p-4+i+j+k+l,2)} \left(\frac{as}{\sigma}, \frac{bs}{\sigma} \right) n_i\, n_j\, n_k\, n_l \end{aligned} \tag{78}$$

where the last line according to eqn (145) of Appendix 1 becomes

$$2 \int_{-\infty}^{\infty} \frac{u^{2p}(au + b)^4}{(1 + u^2)^5}\, \psi\left(u, \frac{as}{\sigma}, \frac{bs}{\sigma} \right) \mathrm{d}u \tag{79}$$

since

$$(-1)^{i+j+k+l}\, u^{i+j+k+l-4}\, n_i\, n_j\, n_k\, n_l = ((-u)^{i-1}\, n_i)^4 = (b + au)^4. \tag{80}$$

In case the shear stress covariance is calculated, we get the same result except that n_j and n_l is replaced by t_j and t_l respectively. Then

$$\begin{aligned}(-1)^{i+j+k+l}\, u^{i+j+k+l-4}\, n_i\, t_j\, n_k\, t_l &= ((-u)^{i-1}\, n_i)^2\, ((-u)^{j-1}\, t_j)^2 \\ &= (b + au)^2\, (a - bu)^2. \end{aligned} \tag{81}$$

Finally, if the covariance between the normal stress at I and the shear stress at II is calculated, solely n_l should be changed to t_l. Thus we have

$$\left.\begin{array}{l}\mathrm{Cov}[\sigma^I_{ij}\, n_i\, n_j, \sigma^{II}_{kl}\, n_k\, n_l] \\ \mathrm{Cov}[\sigma^I_{ij}\, n_i\, n_j, \sigma^{II}_{kl}\, n_k\, t_l] \\ \mathrm{Cov}[\sigma^I_{ij}\, n_i\, t_j, \sigma^{II}_{kl}\, n_k\, t_l]\end{array}\right\} = \frac{2}{\sigma^2} \sum_{p=0}^{2} \int_{\tau_1=0}^{t_1} \int_{\tau_2=0}^{t_2} A_{2p}(\tau_1, \tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

$$\times \left\{ \begin{array}{ll} \displaystyle\int_{-\infty}^{\infty} \frac{u^{2p}(au + b)^4}{(1 + u^2)^5}\, \psi\left(u, \frac{as}{\sigma}, \frac{bs}{\sigma} \right) \mathrm{d}u & (82) \\ \displaystyle\int_{-\infty}^{\infty} \frac{u^{2p}(au + b)^3\, (a - bu)}{(1 + u^2)^2}\, \psi\left(u, \frac{as}{\sigma}, \frac{bs}{\sigma} \right) \mathrm{d}u & (83) \\ \displaystyle\int_{-\infty}^{\infty} \frac{u^{2p}(au + b)^2\, (a - bu)^2}{(1 + u^2)^5}\, \psi\left(u, \frac{as}{\sigma}, \frac{bs}{\sigma} \right) \mathrm{d}u. & (84) \end{array} \right.$$

We see that the first, eqn (82), and the last, eqn (84), of these integrals with respect to u both are of the form as the function $g(s)$ given by eqn (156) of Appendix 2. The variance of the average normal stress across a part of length L of the cut to time t is ([7], p. 307)

$$\mathrm{Var}\left[\frac{1}{L} \int_0^L (\sigma_{ij}\, n_i\, n_j)\, \mathrm{d}s \right] = \frac{4}{L^2} \sum_{p=0}^{2} B_p(t) \int_0^{L/\sigma} \left(\frac{L}{\sigma} - s \right) g_p(s)\, \mathrm{d}s \tag{85}$$

in which $g_p(s)$ is the function defined by the integral with respect to u in eqn (82), while

$$B_p(t) = \int_{\tau_1=0}^{t} \int_{\tau_2=0}^{t} A_{2p}(\tau_1, \tau_2)\, d\tau_1\, d\tau_2. \tag{86}$$

Using eqns (156), (180) of Appendix 2 we have

$$\mathrm{Var}\left[\frac{1}{L}\int_0^L (\sigma_{ij} n_i n_j)\, ds\right] \propto \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{(B_0(t) + B_1(t)u^2 + B_2(t)u^4)(au+b)^2}{(1+u^2)^4}\, du \frac{\log\frac{L}{\sigma}}{L^2} \tag{87}$$

asymptotically for large L/σ. The right side converges to zero for $L \to \infty$ showing that the normal stress process $\sigma_{ij}\, n_i\, n_j$ is ergodic in quadratic mean with respect to the mean ([7], p. 308). A corresponding result is obtained for the shear stress process $\sigma_{ij}\, n_i\, t_j$. In eqn (87) we simply put $(a - bu)^2$ in place of $(au + b)^2$. As an example, by using the table of Appendix 1, eqn (87) may for $a = 1$, $b = 0$ be written

$$\mathrm{Var}\left[\frac{1}{L}\int_0^L (\sigma_{ij}\, n_i\, n_j)\, ds\right] \propto \frac{1}{16\pi}(B_0(t) + B_1(t) + 5B_2(t))\frac{\log\frac{L}{\sigma}}{L^2}. \tag{88}$$

Equation (87) shows that the variance of the average normal stress across any finite piece of the cut approaches infinity for $\sigma \to 0$. This reflects the fact that eqn (65) does not define a Fourier transform. However, it is interesting to note from eqn (87) that for any L

$$\lim_{\sigma\to 0}\frac{1}{-\log\sigma}\mathrm{Var}\left[\frac{1}{L}\int_0^L (\sigma_{ij}\, n_i\, n_j)\, ds\right] = \frac{1}{L^2}\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{(B_0(t) + B_1(t)u^2 + B_2(t)u^4)\,(au+b)^2}{(1+u^2)^4}\, du \tag{89}$$

independent of the particular covariance function ρ_{σ}, eqn (51), used in the limit operation $\sigma \to 0$. The consequence of this observation is discussed in the introduction, motivating the introduction of a formal process called *logarithmically weakened* second moment white noise.

THE STRAIN FIELD

In order to calculate the covariance function of the strain tensor field for a fixed $\sigma > 0$, the covariance between the two terms of eqn (48) is needed. By forming the covariance between $\epsilon_{ij}(\mathbf{r}_1, t_1)$ as given by eqn (48) and the left side of eqn (55) for $t = t_2$ we get the following equation

$$\mathrm{Cov}\left[\epsilon_{ij}(\mathbf{r}_1, t_1), \int_{\tau_2=0}^{t_2} C(t_2, \tau_2)(\Delta^2)_{\mathbf{r}_2}\, d\Phi(\mathbf{r}_2, \tau_2)\right]$$
$$= -\mathrm{Cov}\left[\int_{\tau_1=0}^{t_1}\int_{\dot{m}(\tau_1)}^{\dot{m}(\tau_1)+d\dot{m}(\tau_1)} S_{ijrs}(\mathbf{r}_1, t_1, \tau_1, \tilde{u})\, du_{rs},\right.$$
$$\left.\int_{\tau_2=0}^{t_2}\int_{\dot{m}(\tau_2)}^{\dot{m}(\tau_2)+d\dot{m}(\tau_2)} (S_{11mn,22} + S_{22mn,11} - 2S_{12mn,12})(\mathbf{r}_2, t_2, \tau_2, \tilde{u})\, du_{mn}\right]$$
$$+ \mathrm{Cov}\left[\int_{\tau_1=0}^{t_1} K_{ijrs}(t_1, \tau_1)\, d\sigma_{rs}(\mathbf{r}_1, \tau_1), \int_{\tau_2=0}^{t_2} C(t_2, \tau_2)(\Delta^2)_{\mathbf{r}_2}\, d\Phi(\mathbf{r}_2, \tau_2)\right]. \tag{90}$$

Defining

$$G_{ij}(\mathbf{r}_2 - \mathbf{r}_1, t_1, t_2) = \int_{\tau_2=0}^{t_2} C(t_2, \tau_2)\,\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \mathrm{d}\Phi(\mathbf{r}_2, \tau_2)] \tag{91}$$

the left side of eqn (90) may be written as $\Delta^2 G_{ij}(\mathbf{r}_2 - \mathbf{r}_1, t_1, t_2)$. With K_{ijrs} given by eqn (40) the second term on the right side of eqn (90) becomes

$$\Delta^2 D_{ij} F(\mathbf{r}_2 - \mathbf{r}_1, t_1, t_2) \tag{92}$$

in which F is defined by eqn (58) and D_{ij} is the differential operator

$$D_{ij} = -\nu\delta_{ij}\Delta + (1+\nu)(-1)^{i+j}\frac{\partial^2}{\partial x_1^{i+j-2}\partial x_2^{4-i-j}}. \tag{93}$$

The first term on the right side of eqn (90) is

$$-\frac{1}{\sigma^2}\sum_{\alpha=0}^{2} b_{ij\alpha}(t_1, t_2)\frac{\mu^2}{\partial x_1^\alpha\,\partial x_2^{2-\alpha}}\rho\left(\frac{x_1^2 + x_2^2}{\sigma^2}\right) \tag{94}$$

in which

$$b_{ij0} = \int_0^{t_1} d_{ij11},\ b_{ij1} = -2\int_0^{t_1} d_{ij12},\ b_{ij2} = \int_0^{t_1} d_{ij22} \tag{95}$$

are given functions of (t_1, t_2) ($t_1 \le t_2$, see definition of d_{ijkl} at eqn (60)). Thus eqn (90) may be written as

$$\Delta^2 H_{ij} = -\frac{1}{\sigma^2}\sum_{\alpha=0}^{2} b_{ij\alpha}\frac{\partial^2}{\partial x_1^\alpha\,\partial x_2^{2-\alpha}}\rho\left(\frac{x_1^2 + x_2^2}{\sigma^2}\right) \tag{96}$$

$$H_{ij} = G_{ij} - D_{ij}F. \tag{97}$$

The Fourier transform of H_{ij} becomes

$$\hat{H}_{ij}(\omega_1, \omega_2) = \sum_{\alpha=0}^{2} b_{ij\alpha}\frac{\omega_1^\alpha\omega_2^{2-\alpha}}{(\omega_1^2 + \omega_2^2)^2}\hat{\rho}((\sigma\omega_1)^2 + (\sigma\omega_2)^2). \tag{98}$$

Since, however, we need the covariance $\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \mathrm{d}\sigma_{kl}(\mathbf{r}_2, \tau_2)]$ rather than that of the integrand of eqn (91), it follows from eqns (52) and (91) that our interest is more in

$$(-1)^{k+1} G_{ij,3-k\,3-l} = \int_{\tau_2=0}^{t_2} C(t_2, \tau_2)\,\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \mathrm{d}\sigma_{kl}(\mathbf{r}_2, \tau_2)] \tag{99}$$

than in G_{ij} itself. Referring to eqn (97) we thus need to invert the Fourier transform

$$-\omega_1^{k+l-2}\,\omega_2^{4-k-l}\,\hat{H}_{ij}(\omega_1, \omega_2) \tag{100}$$

to get $H_{ij,3-k\,3-l}$. Comparison of eqn (71) with eqn (100) after substitution of eqn (98) shows that

$$H_{ij,3-k\,3-l}(x_1, x_2) = -\frac{1}{\sigma^2}\sum_{\alpha=0}^{2} b_{ij\alpha}F_{(\alpha+k+l-2,1)}\left(\frac{x_1}{\sigma}, \frac{x_2}{\sigma}\right). \tag{101}$$

Equation (97) then gives

$$G_{ij,3-k\,3-l}(x_1, x_2, t_1, t_2) = H_{ij,3-k\,3-l}(x_1, x_2, t_1, t_2) + D_{ij}F_{,3-k\,3-l} \qquad (102)$$

in which, eqns (93) and (72),

$$\begin{aligned} D_{ij}F_{,3-k\,3-l} &= -\nu\delta_{ij}\left(\frac{\partial^4 F}{\partial x_1^{k+l}\partial x_2^{4-k-l}} + \frac{\partial^4 F}{\partial x_1^{k+l-2}\partial x_2^{6-k-l}}\right) \\ &\quad + (1+\nu)(-1)^{i+j}\frac{\partial^4 F}{\partial x_1^{i+j+k+l-4}\partial x_2^{8-i-j-k-l}} \\ &= \frac{1}{\sigma^2}\sum_{p=0}^{2} a_{2p}[-\nu\delta_{ij}(F_{(2p+k+l,2)} + F_{(2p+k+l-2,2)}) \\ &\quad + (1+\nu)(-1)^{i+j}F_{(2p+i+j+k+l-4,2)}]. \end{aligned} \qquad (103)$$

Thus eqns (99), (102), (101), and (103) give

$$\int_{\tau_2=0}^{t_2} C(t_2,\tau_2)\,\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1,t_1), \mathrm{d}\sigma_{kl}(\mathbf{r}_2,\tau_2)] = (-1)^{k+l}(\text{sum of the right sides of eqns 101 and 103}). \qquad (104)$$

In order to calculate the covariance $\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \epsilon_{kl}(\mathbf{r}_2, t_2)]$ from eqn (48) we note that it is a problem of the following type. Let $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ be random variables such that $Z_1 = X_1 + Y_1$ and $Z_2 = X_2 + Y_2$. Then from expanding the right side of the identity $\mathrm{Cov}[X_1, X_2] = \mathrm{Cov}[Z_1 - Y_1, Z_2 - Y_2]$ and solving with respect to $\mathrm{Cov}[Z_1, Z_2]$ it follows that

$$\mathrm{Cov}[Z_1,Z_2] = \mathrm{Cov}[X_1,X_2] - \mathrm{Cov}[Y_1,Y_2] + \mathrm{Cov}[Z_1,Y_2] + \mathrm{Cov}[Y_1,Z_2]. \qquad (105)$$

Using this on eqn (48) we get

$$\begin{aligned} \mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1,t_1), \epsilon_{kl}(\mathbf{r}_2,t_2)] &= \mathrm{Cov}\Bigg[\int_{\tau_1=0}^{t_1}\int_{\dot m(\tau_1)}^{\dot m(\tau_1)+\mathrm{d}\dot m(\tau_1)} S_{ijmn}(\mathbf{r}_1,t_1,\tau_1,\bar u)\,\mathrm{d}u_{mn}, \\ &\quad \int_{\tau_2=0}^{t_2}\int_{\dot m(\tau_2)}^{\dot m(\tau_2)+\mathrm{d}\dot m(\tau_2)} S_{klrs}(\mathbf{r}_2,t_2,\tau_2,\bar u)\,\mathrm{d}u_{rs}\Bigg] \\ &\quad - \int_{\tau_1=0}^{t_1}\int_{\tau_2=0}^{t_2} K_{ijmn}(t_1,\tau_1)K_{klrs}(t_2,\tau_2) \\ &\quad \times \mathrm{Cov}[\mathrm{d}\sigma_{mn}(\mathbf{r}_1,\tau_1), \mathrm{d}\sigma_{rs}(\mathbf{r}_2,\tau_2)] \\ &\quad + \int_{\tau_2=0}^{t_2} K_{klrs}(t_2,\tau_2)\,\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1,t_1), \mathrm{d}\sigma_{rs}(\mathbf{r}_2,\tau_2)] \\ &\quad + \int_{\tau_1=0}^{t_1} K_{ijmn}(t_1,\tau_1)\,\mathrm{Cov}[\epsilon_{kl}(\mathbf{r}_2,t_2), \mathrm{d}\sigma_{mn}(\mathbf{r}_1,\tau_1)] \end{aligned} \qquad (106)$$

$$\begin{aligned} &= \frac{1}{\sigma^2}\rho\left(\frac{x_1^2+x_2^2}{\sigma^2}\right)\int_0^{t_1} d_{ijkl}\,d\tau \\ &\quad - \int_{\tau_1=0}^{t_1}\int_{\tau_2=0}^{t_2} C_1C_2[\nu_1\nu_2\,\delta_{ij}\delta_{kl}\,\mathrm{Cov}[\mathrm{d}\sigma_{ss}, \mathrm{d}\sigma_{ss}] \\ &\quad - \nu_1(1+\nu_2)\delta_{ij}\,\mathrm{Cov}[\mathrm{d}\sigma_{ss}, \mathrm{d}\sigma_{kl}] - \nu_2(1+\nu_1)\delta_{kl}\,\mathrm{Cov}[\mathrm{d}\sigma_{ij}, \mathrm{d}\sigma_{ss}] \\ &\quad + (1+\nu_1)(1+\nu_2)\,\mathrm{Cov}[\mathrm{d}\sigma_{ij}\ \mathrm{d}\sigma_{kl}]]\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \end{aligned}$$

$$+ \int_{\tau_1=0}^{t_1} C_1[-\nu_1\delta_{ij}\,\mathrm{Cov}[\mathrm{d}\sigma_{ss}, \epsilon_{kl}] + (1+\nu_1)\,\mathrm{Cov}[\mathrm{d}\sigma_{ij}, \epsilon_{kl}]]\,\mathrm{d}\tau_1$$

$$+ \int_{\tau_2=0}^{t_2} C_2[-\nu_2\delta_{kl}\,\mathrm{Cov}[\epsilon_{ij}, \mathrm{d}\sigma_{ss}] + (1+\nu_2)\,\mathrm{Cov}[\epsilon_{ij}, \mathrm{d}\sigma_{kl}]]\,\mathrm{d}\tau_2 \tag{107}$$

in which the notation has been shortened conveniently. By use of eqns (73), (75), (76), (104) and of eqn (145) of Appendix 1 the covariances of the integrands of eqn (107) may next be calculated. Substantial simplifications show up in case the Poisson ratio function $\nu(t, \tau)$ is a constant. In the following we will assume that this is the case, i.e. that $\nu_1 = \nu_2 = \nu$. Then we get

$$\sigma^2\,\mathrm{Cov}[\epsilon_{ij}, \epsilon_{kl}] = \rho\left(\frac{x_1^2 + x_2^2}{\sigma^2}\right)\int_0^{t_1} d_{ijkl}\,\mathrm{d}\tau$$
$$+ \nu^2\delta_{ij}\delta_{kl}(a_0F_{(0,2)} + (2a_0 + a_2)F_{(2,2)} + (a_0 + 2a_2 + a_4)F_{(4,2)}$$
$$+ (a_2 + 2a_4)F_{(6,2)} + a_4F_{(8,2)}) + (1+\nu)^2(-1)^{i+j+k+l}$$
$$\times [a_0F_{(i+j+k+l-4,2)} + a_2F_{(i+j+k+l-2,2)} + a_4\mathrm{s}F_{(i+j+k+l,2)}]$$
$$-\nu(1+\nu)(-1)^{k+l}\delta_{ij}[a_0F_{(k+l-2,2)} + (a_0 + a_2)F_{(k+l,2)}$$
$$+ (a_2 + a_4)F_{(k+l+s2,2)} + a_4F_{(k+l+4,2)}]$$
$$-\nu(1+\nu)(-1)^{i+j}\delta_{kl}[a_0F_{(i+j-2,2)} + (a_0 + a_2)F_{(i+j,2)}$$
$$+ (a_2 + a_4)F_{(i+j+2,2)} + a_4F_{(i+j+4,2)}]$$
$$+\nu\delta_{kl}[b_{ij0}(F_{(0,1)} + F_{(2,1)}) + b_{ij1}(F_{(1,1)} + F_{(3,1)}) + b_{ij2}(F_{(2,1)} + F_{(4,1)})]$$
$$+\nu\delta_{ij}[b_{kl0}(F_{(0,1)} + F_{(2,1)}) + b_{kl1}(F_{(1,1)} + F_{(3,1)}) + b_{kl2}(F_{(2,1)} + F_{(4,1)})]$$
$$-(1+\nu)(-1)^{k+l}[b_{ij0}F_{(k+l-2,1)} + b_{ij1}F_{(k+l-1,1)} + b_{ij2}F_{(k+l,1)}]$$
$$-(1+\nu)(-1)^{i+j}[b_{kl0}F_{(i+j-2,1)} + b_{kl1}F_{(i+j-1,1)} + b_{kl2}F_{(i+j,1)}]. \tag{108}$$

By use of eqn (145) of Appendix 1 and that $\rho(x_1^2 + x_2^2) = F_{(0,0)}(x_1, x_2)$, this gives

$$\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1),\ \epsilon_{kl}(\mathbf{r}_2, t_2)]$$
$$= \frac{2}{\sigma^2}\int_{-\infty}^{\infty}\frac{\psi\left(u, \frac{x_1}{\sigma}, \frac{x_2}{\sigma}\right)}{(1+u^2)^5}\Bigg\{(1+u^2)^4\int_0^{t_1} d_{ijkl}\,\mathrm{d}\tau + [\nu^2\delta_{ij}\delta_{kl}(1+u^2)^2$$
$$+ (1+\nu)^2(-u)^{i+j+k+l-4} - \nu(1+\nu)((-u)^{k+l-2}\delta_{ij} + (-u)^{i+j-2}\delta_{kl})(1 + u^2)]$$
$$\times (a_0 + a_2u^2 + a_4u^4) + (1+u^2)^2[(\nu\delta_{kl}(1+u^2) - (1+\nu)(-u)^{k+l-2})$$
$$\times (b_{ij0} + b_{ij1}u + b_{ij2}u^2) + (\nu\delta_{ij}(1+u^2) - (1+\nu)(-u)^{i+j-2})$$
$$\times (b_{kl0} + b_{kl1}u + b_{kl2}u^2)]\Bigg\}\,\mathrm{d}u \tag{109}$$

DISPLACEMENTS RELATIVE TO THE ORIGIN

The displacement relative to the origin O in the radial direction of a point P of distance L from the origin is

$$u_{OP}(L) = \int_0^{L/\sigma} \epsilon_{ij}(as, bs)\,t_it_j\,\mathrm{d}s \tag{110}$$

in which $(t_1, t_2) = (a, b)$ is a unit vector in the direction OP. The notation $\epsilon_{ij}(as, bs)$

is short for $\epsilon_{ij}((x_1, x_2), t)$ with $x_1 = as$, $x_2 = bs$. The variance is

$$\mathrm{Var}[u_{OP}(L)] = 2\int_0^{L/\sigma}\left(\frac{L}{\sigma} - s\right)\mathrm{Cov}[\epsilon_{ij}(0, 0), \epsilon_{kl}(as, bs)]\, t_i t_j t_k t_l\, \mathrm{d}s \tag{111}$$

where $\mathrm{Cov}[\epsilon_{ij}(0, 0), \epsilon_{kl}(as, bs)]$ is given by eqn (109) for $\sigma = 1$. Using eqns (41) and (62) to calculate the factor to $\rho(r^2)$ in eqn (109), we have

$$\begin{aligned}
&\mathrm{Cov}[\epsilon_{ij}(0,0), \epsilon_{kl}(as, bs)]\, t_i t_j t_k t_l\\
&= 2\int_{-\infty}^{\infty}\frac{\psi(u, as, bs)}{(1+u^2)^5}\{(1+u^2)^4(a_0a^4 + a_2a^2b^2 + a_4b^4) + (\nu(1+u^2) - (1+\nu)\\
&\times (a - bu)^2)^2(a_0 + a_2u^2 + a_4u^4) + 2(1+u^2)^2(\nu(1+u^2) - (1+\nu)(a-bu)^2)\\
&\times (c_0 + c_1u + c_2u^2)\}\mathrm{d}u
\end{aligned} \tag{112}$$

in which

$$\begin{aligned}
c_0 &= a_0a^2 + b_{220}b^2\\
c_1 &= 2b_{121}ab\\
c_2 &= b_{112}a^2 + a_4b^2.
\end{aligned} \tag{113}$$

The right side of eqn (111) gets the form

$$\begin{aligned}
&4\int_{-\infty}^{\infty}R(u)\int_0^{\lambda}(\lambda - s)\,\psi(u, as, bs)\,\mathrm{d}s\,\mathrm{d}u\\
&\quad = 4\int_{-\infty}^{\infty}R(u)\int_0^{\lambda}(\lambda - s)\int_0^{\infty}v\,\hat{\rho}(v^2)\cos\left(\frac{au + b}{\sqrt{1+u^2}}sv\right)\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}u
\end{aligned} \tag{114}$$

in which $\lambda = L/\sigma$, and $R(u)$ is a bounded rational function of order of magnitude $O(1/u^2)$ for $|u| \to \infty$. It is shown in Appendix 2 that this expression under quite general conditions on $\hat{\rho}(v^2)$ asymptotically equals, eqn (193),

$$\frac{4\pi}{a^2}R\left(-\frac{b}{a}\right)\left(\int_0^{\infty}\hat{\rho}(v^2)\,\mathrm{d}v\right)\frac{L}{\sigma} \tag{115}$$

for large L/σ. Thus we get

$$\begin{aligned}
\mathrm{Var}[u_{OP}(L)] &\propto 8\pi(ab)^2(a_2 - b_{220} - b_{112} - 2b_{121})\left(\int_0^{\infty}\hat{\rho}(v^2)\,\mathrm{d}v\right)\frac{L}{\sigma}\\
&= 64\pi(ab)^2\left(\int_0^{t}d_{1212}(t, \tau)\,\mathrm{d}\tau\right)\left(\int_0^{\infty}\hat{\rho}(v^2)\,\mathrm{d}v\right)\frac{L}{\sigma}
\end{aligned} \tag{116}$$

asymptotically for large L/σ. The last step follows by using that $b_{220} = b_{112}$ and $a_2 = 2(b_{112} - b_{121})$, eqns (62), (95). Given that $d_{1212}(t, \tau)$ is not zero, the implication is that if the factor $1/\sigma^2$ of eqn (51) is changed to $1/\sigma$ and we pass to the limit $\sigma \to 0$, then the variances of the radial relative displacements become proportional to L. The variance is zero for $ab = 0$, i.e. in the directions of the principal axes of the mean stress tensor, and it attains its maximum for $a = b = 1/\sqrt{2}$. The resulting white noise process of zero intensity may be called a *linearly weakened* second moment white noise process. For this model the average normal stress or shear stress across any linear cut of finite or infinite length is deterministic since $\sigma/\log\sigma \to 0$ for $\sigma \to 0$, eqn (87).

For $a = 1$ the rational function $R(u)$ becomes

$$R(u) = [u^2(2 + \nu u^2)^2 a_0 + (\nu u^2 - 1)^2(a_2 + a_4 u^2) + 2(1 + u^2)^2(\nu u^2 - 1)b_{112}]\, u^2/(1 + u^2)^5. \quad (117)$$

Due to the factor u^2 it may from the investigations of the Appendix leading to the limit result of eqn (180) be concluded that

$$\mathrm{Var}[u_{OP}(L)] \propto \frac{1}{\pi^2}\left(\int_{-\infty}^{\infty} R(u)\,\frac{1 + u^2}{u^2}\,\mathrm{d}u\right)\log\frac{L}{\sigma} \quad (118)$$

asymptotically for large L/σ and valid for the relative displacements in direction of the x_1 axis of two points O and P on the x_1 axis and in mutual distance L. It follows from eqns (118) and (116) that the variance will be finite and constant for the logarithmically weakened white noise model in case $a = 1$ (or $b = 1$) but infinite for all other directions.

If b_{121} is zero, i.e. if $d_{1212}(t, \tau)$ is zero identically, $\mathrm{Var}[u_{OP}(L)]$ for any direction increases with L/σ at less order of magnitude than L/σ. For $a = 1$ or $b = 1$ the order is $\log L/\sigma$. It seems rather difficult to find the exact order of magnitude for other directions.

SCALAR PROCESS CREEP

Consider the special case of the scalar process creep of Example 3 leading to eqn (46). Assume that both ν and $a = w/c$ are time independent constants and that the expected stress tensor increment $\mathrm{d}\bar{m}$ takes place at time zero, i.e. $\mathrm{d}m_{ij}(\tau) = \delta(\tau)\,\mu_{ij}$, with μ_{ij} being a constant *principal* stress tensor. Then the integrands of eqn (62) become

$$\frac{d_{1111}}{h_0} = \frac{2[d_{1122} + 2d_{1212})}{h_2} = \frac{d_{2222}}{h_4} = \frac{a\,C(t,\tau)\delta(\tau)}{\sqrt{\mu_{11}^2 + \mu_{22}^2}} \quad (119)$$

with

$$h_0 = (\mu_{11} - \nu\mu_{22})^2, \quad h_4 = (\mu_{22} - \nu\mu_{11})^2 \quad (120)$$

$$h_2 = 2(\mu_{11} - \nu\mu_{22})(\mu_{22} - \nu\mu_{11}) \quad (121)$$

and we have

$$a_{2p}(t_1, t_2) = k_{2p}\int_{0-}^{t_1} \delta(\tau)\,C(t_1, \tau) = k_{2p}\,C(t_1, 0) \quad (122)$$

in which

$$k_{2p} = \frac{a}{\sqrt{\mu_{11}^2 + \mu_{22}^2}}\,h_{2p}. \quad (123)$$

The integral equation, eqn (76), in this case gets the form

$$\int_{\tau_1=0-}^{t_1} C(t_1, \tau_1)\left[\int_{\tau_2=0-}^{t_1} C(t_2, \tau_2)A_{2p}(\tau_1, \tau_2)\,\mathrm{d}\tau_2\right]\mathrm{d}\tau_1 = k_{2p}\,C(t_1, 0) \quad (124)$$

from which it follows that

$$\int_{\tau_2=0}^{t_2} C(t_2, \tau_2)\,A_{2p}(\tau_1, \tau_2)\,\mathrm{d}\tau_2 = k_{2p}\,\delta(\tau_1). \quad (125)$$

Given that the homogeneous integral equation obtained from eqn (125) by writing 0 in place of $k_{2p}\,\delta(\tau_1)$ has only the zero function as solution, we may write the unique solution to eqn (125) on the form

$$A_{2p}(\tau_1, \tau_2) = k_{2p}\,\delta(\tau_1)\,R(\tau_2) \quad (126)$$

where $R(\tau)$ is the solution to the integral equation

$$\int_0^t C(t,\tau)\,R(\tau)\,\mathrm{d}\tau = 1. \tag{127}$$

It is seen that $R(\tau)$ is the so-called relaxation function corresponding to the creep function $C(t,\tau)$ for a unit strain increment applied to time zero and kept constant thereafter.

The covariance function of the stress tensor, eqn (77), becomes

$$\mathrm{Cov}[\sigma_{ij}(\mathbf{r}_1,t_1),\sigma_{kl}(\mathbf{r}_2,t_2)] = \frac{2a}{\sigma^2}\int_{\tau=0}^{t_1} R(\tau)\,\mathrm{d}\tau \int_{-\infty}^{\infty} \frac{(-u)^{i+j+k+l}[\mu_{11}-\nu\mu_{22}+(\mu_{22}-\nu\mu_{11})u^2]^2}{u^4(1+u^2)^5\sqrt{\mu_{11}^2+\mu_{22}^2}} \times \psi\left(u,\frac{x_1}{\sigma},\frac{x_2}{\sigma}\right)\mathrm{d}u. \tag{128}$$

Plots of the component correlation functions as functions of x_1/σ, x_2/σ are shown in Fig. 2. They correspond to the particular cases of the correlation functions of eqns (66) and (67) for which the function $\psi(u,x_1,x_2)$ is given in Appendix 3 by eqns (203) and (208) respectively.

The asymptotic result, eqn (87), for the average normal stress on a linear cut of length L and normal vector (n_1,n_2) becomes

$$\mathrm{Var}\left[\frac{1}{L}\int_0^L(\sigma_{ij}\,n_i\,n_j)\,\mathrm{d}s\right]\frac{L^2}{\log\frac{L}{\sigma}}$$

$$\propto \frac{a}{\pi^2}\int_{\tau=0}^{t} R(\tau)\,\mathrm{d}\tau\int_{-\infty}^{\infty}\frac{[\mu_{11}-\nu\mu_{22}+u^2(\mu_{22}-\nu\mu_{11})]^2(n_1-n_2u)^2}{(1+u^2)^4\sqrt{\mu_{11}^2+\mu_{22}^2}}$$

$$= \frac{\pi a}{16\sqrt{\mu_{11}^2+\mu_{22}^2}}\left[(1-\nu)^2(\mu_{11}+\mu_{22})^2+\begin{Bmatrix}4(\mu_{11}-\nu\mu_{22})^2\\ 2(\mu_{11}-\nu\mu_{22})^2+2(\mu_{22}-\nu\mu_{11})^2\\ 4(\mu_{22}-\nu\mu_{11})^2\end{Bmatrix}\right] \times\int_{\tau=0}^{t} R(\tau)\,\mathrm{d}\tau \tag{129}$$

where the last three expressions are valid for $n_1 = 1$, $n_1 = 1/\sqrt{2}$, $n_1 = 0$ respectively.

For the strain tensor we get eqn (109) with

$$\int_0^{t_1} d_{ijkl}\,\mathrm{d}\tau = \int_{\tau=0}^{t_1}\text{(integrand defined by eqns (42), (46), (47) for } \mathrm{d}\sigma_{ij} = \mathrm{d}m_{ij}(\tau))$$

$$= [(1+\nu)^2\mu_{ij}\mu_{kl} - \nu(1+\nu)(\delta_{ij}\mu_{kl}+\delta_{kl}\mu_{ij})\mu_{ss}+\nu^2(\mu_{ss})^2\delta_{ij}\delta_{kl}] \times\frac{aC(t_1,0)}{\sqrt{\mu_{11}^2+\mu_{22}^2}} \tag{130}$$

substituted. Further we have for substitution into eqn (109):

$$b_{ij0}(t_1,t_2) = [(1+\nu)^2\mu_{11}\mu_{ij}-\nu(1+\nu)(\mu_{11}+\nu_{22})(\mu_{11}\delta_{ij}+\mu_{ij})+\nu^2(\mu_{11}+\mu_{22})^2\delta_{ij}]\frac{aC(t_1,0)}{\sqrt{\mu_{11}^2+\mu_{22}^2}} \tag{131}$$

$$b_{ij2} = b_{ij0}\ \text{with interchange of}\ \mu_{11}\ \text{and}\ \mu_{22} \tag{132}$$

$$b_{ij1} = 0. \tag{133}$$

With respect to the results of the previous section we note that $\mathrm{Var}[u_{OP}(L)]$ increases with L/σ at less order of magnitude than L/σ. For the uniaxial case $\mu_{22} = 0$ we get

$$a_0 = |\mu_{11}|aC,\ a_2 = -2\nu a_0,\ a_4 = \nu^2 a_0 \tag{134}$$

$$\frac{1}{a_0}\int_0^{t_1} d_{ijkl}\,\mathrm{d}\tau = \begin{cases} 1 & \text{for } d_{1111}\\ -\nu & \text{for } d_{1122} \text{ or } d_{2211}\\ \nu^2 & \text{for } d_{2222}\\ 0 & \text{otherwise}\end{cases} \tag{135}$$

$$b_{110} = a_0,\ b_{220} = b_{112} = -\nu a_0,\ b_{222} = \nu^2 a_0$$

$$b_{ij0} = b_{ij2} = 0 \quad \text{for} \quad i \neq j \tag{136}$$

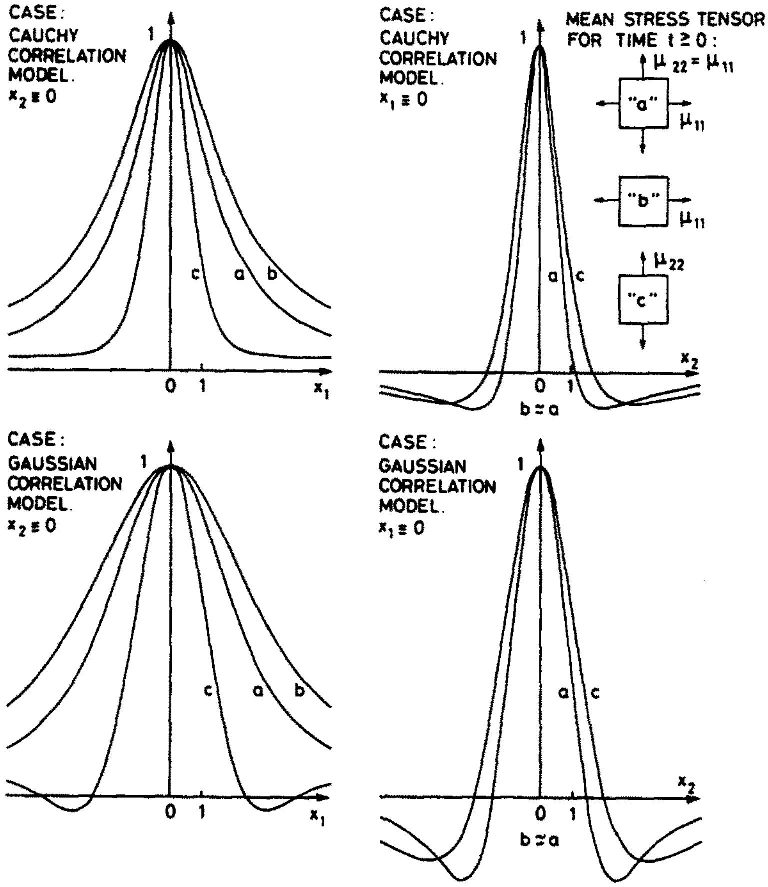

Fig. 2. Correlation functions $\rho[\sigma_{11}(0, 0, t), \sigma_{11}(x_1, x_2, t)]$ corresponding to scalar process creep and the strain correlation model of Cauchy type (eqn (67)) and of Gaussian type (eqn (66)) for $\sigma = 1$. As indicated in the upper right corner, the curves "a" correspond to the case $\mu_{11} = \mu_{22}$ (independent of value of Poisson's ratio ν), the curves "b" correspond to $\mu_{22} = 0$, $\nu = \frac{1}{6}$, and the curves "c" correspond to $\mu_{11} = 0$, $\nu = \frac{1}{6}$, all applied to time $t = 0$. For the case $x_1 = 0$, curve "b" is very close to "a".

giving the variances

$$\mathrm{Var}[\epsilon_{11}]\frac{\sigma^2}{\rho(0)\,|\,\mu_{11}\,|\,aC} = \frac{1}{\pi}(1+\nu)^2\int_{-\infty}^{\infty}\frac{4u^4 + 4(1-\nu)u^6 + (1-\nu)^2u^8}{(1+u^2)^5}\,\mathrm{d}u$$

$$= \frac{1}{128}(35(1-\nu)^2 + 20(1-\nu) + 12)(1+\nu)^2 = 0.643^2 \tag{137}$$

$$\mathrm{Var}[\epsilon_{22}]\frac{\sigma^2}{\rho(0)\,|\,\mu_{11}\,|\,aC} = \frac{1}{\pi}(1+\nu)^4\int_{-\infty}^{\infty}\frac{u^4}{(1+u^2)^5}\,\mathrm{d}u$$

$$= \frac{3}{128}(1+\nu)^4 = 0.208^2 \tag{138}$$

$$\mathrm{Var}[\epsilon_{12}]\frac{\sigma^2}{\rho(0)\,|\,\mu_{11}\,|\,aC} = \frac{1}{\pi}(1+\nu)^2\int_{-\infty}^{\infty}\frac{u^2 - 2\nu u^4 + \nu^2u^6}{(1+u^2)^5}\,\mathrm{d}u$$

$$= \frac{1}{128}(5\nu^2 - 6\nu + 5)(1+\nu)^2 = 0.210^2 \tag{139}$$

where the last numbers correspond to the value $\nu = \frac{1}{6}$ which is typical for concrete. If the compatibility condition is not taken into account, only the first term in the bracket of eqn (109) is present. In that case the above variance factors for ϵ_{11}, ϵ_{22} and ϵ_{12} are 1, ν^2 and 0 respectively. Thus the standard deviation of ϵ_{11} is for $\nu = \frac{1}{6}$ decreased by a factor of 0.64 due to compatibility. The standard deviation of ϵ_{22} is increased by the factor 1.25 (= 6 × 0.208) while compatibility gives approximately the same standard deviation of ϵ_{22} and ϵ_{12}, the last being zero under neglection of compatibility.

SUMMARY AND CONCLUSIONS

The model of statistically isotropic visco-elastic stochastic creep formulated in the previously published first part [6] of this paper is considered herein for the case of a two-dimensional model. A stochastically homogeneous random stress field extended over the entire plane is described in terms of its covariance structure as function of the spatially constant mean stress increments in time given that the coordinate system of principal mean stresses is time invariant. In order to obtain this stress field solution to the compatibility equation and the local equilibrium equations in terms of a random stress function and under consideration of the linearized stochastic constitutive equations as they are formulated in Ref. [6], it is necessary to change the formulation of the strain field spatial correlation for given stress increment from being modelled in terms of a Dirac delta function to be modelled in terms of an element in a sequence of covariance functions that in the limit is a Dirac delta function. The covariance function corresponds to an index parameter σ such that the Dirac delta function is obtained in the limit $\sigma \to 0$. Fourier transform technique is applied in the solution procedure. Interesting asymptotic features of the solution as $\sigma \to 0$ may be studied. For example, the average normal stress on any linear cut of length L has its variance proportional to $\log (L/\sigma)/L^2$ for large L/σ. This means that if the covariance functions of the sequence are weakened by division by $\log (1/\sigma)$ then the standard deviation of the average stress on any linear cut of length L becomes finite and proportional to $1/L$ in the limit $\sigma \to 0$. For a body with given deterministic external stress distribution such a logarithmically weakened second moment white noise model gives a deterministic strain tensor in average across the body. Thus this type of stochastic creep is not observable by use of standard tests applying external statically determinate test arrangements.

The solution for the strain field is considerably more complicated than that for the stress field. A study of the displacements relative to the origin is particularly interesting. For large L/σ, the variance of the radial relative displacement of a point in distance L from the origin is in general proportional to L/σ or it increases slower than L/σ but at least as fast as $\log (L/\sigma)$. In the direction of the mean stress principal axes the variance is proportional to $\log (L/\sigma)$ for large L/σ.

A particular simple type of isotropic stochastic creep is scalar process creep (more restrictively called Poisson process viscous creep in Ref. 6). The results herein are specialized to this case for a mean stress tensor increment taking place at time zero and kept constant thereafter. For the scalar process creep the variance of the radial relative displacement increases slower than L/σ for large L/σ.

Appendices 1–3 contain some purely mathematical results concerning particular Fourier transforms related to the Laplace operator and covariance functions of two-dimensional isotropic random fields. These results may be useful also in other contexts: eqns (145), (146), (152), (172), (180), (193), (195), (203), (208).

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APPENDIX 1

Inversion of Fourier transform

With α and β being non-negative integers such that $0 \le \alpha \le 4\beta$ the inverse Fourier transform of

$$\hat{F}_{(\alpha,\beta)}(\omega_1,\omega_2) = \frac{\omega_1^{\alpha}\omega_2^{4\beta-\alpha}}{(\omega_1^2+\omega_2^2)^{2\beta}}\,\hat{\rho}(\omega_1^2+\omega_2^2) \tag{140}$$

is given by

$$\begin{aligned}
&F_{(\alpha,\beta)}(x_1,x_2)\\
&= \lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} d\omega_2 \int_{-\lambda}^{\lambda} \frac{\omega_1^{\alpha}\omega_2^{4\beta-\alpha}}{(\omega_1^2+\omega_2^2)^{2\beta}}\,\hat{\rho}(\omega_1^2+\omega_2^2)\, e^{i(x_1\omega_1+x_2\omega_2)}\, d\omega_1\\
&= \lim_{\lambda\to\infty}\Bigg[\int_{-\lambda}^{\lambda} |\omega_2| \left(\frac{\omega_2}{|\omega_2|}\right)^{\alpha} e^{ix_2\omega_2}\, d\omega_2 \int_{-1}^{1}\frac{u^{\alpha}}{(1+u^2)^{2\beta}}\\
&\times \hat{\rho}[\omega_2^2(1+u^2)]\, e^{ix_1|\omega_2|u}\, du\\
&+ \int_{-\lambda}^{\lambda} |\omega_1| \left(\frac{\omega_1}{|\omega_1|}\right)^{\alpha} e^{ix_1\omega_1}\, d\omega_1 \int_{-1}^{1}\frac{u^{4\beta-\alpha}}{(1+u^2)^{2\beta}}\,\hat{\rho}[\omega_1^2(1+u^2)]\\
&\times e^{ix_2|\omega_1|u}\, du\Bigg]\\
&= \lim_{\lambda\to\infty}\Bigg[\int_{-1}^{1}\frac{u^{\alpha}}{(1+u^2)^{2\beta}}\, du \int_{-\lambda}^{\lambda}|\omega|\,\hat{\rho}[\omega^2(1+u^2)]\, e^{i(x_2+ux_1)\omega}\, d\omega\\
&+ \int_{-1}^{1}\frac{u^{4\beta-\alpha}}{(1+u^2)^{2\beta}}\, du \int_{-\lambda}^{\lambda}|\omega|\,\hat{\rho}[\omega^2(1+u^2)]\, e^{i(x_1+ux_2)\omega}\, d\omega\Bigg].
\end{aligned} \tag{141}$$

Assume that the nonnegative function $\hat{\rho}$ satisfies the condition

$$\hat{\rho}(\omega^2) \le K\min\{\omega^{-2+\epsilon}, \omega^{-2-\epsilon}\} \tag{142}$$

for some positive constants K and ϵ. We then have

$$\begin{aligned}
&\left|\int_{-\lambda}^{\lambda}|\omega|\,\hat{\rho}[\omega^2(1+u^2)]\, e^{i(x_2+ux_1)\omega}\, d\omega\right|\\
&\le 2\int_0^{\lambda}\omega\,\hat{\rho}[\omega^2(1+u^2)]\, d\omega\\
&\le \frac{K}{1+u^2}\Bigg[(1+u^2)^{\epsilon/2}\int_0^{1/\sqrt{1+u^2}}\omega^{-1+\epsilon}\, d\omega\\
&+ (1+u^2)^{-\epsilon/2}\int_{1/\sqrt{1+u^2}}^{\lambda}\omega^{-1-\epsilon}\, d\omega\Bigg]\\
&= \frac{K}{\epsilon(1+u^2)}\,[2-(\lambda\sqrt{1+u^2})^{-\epsilon}] < \frac{2K}{\epsilon(1+u^2)}
\end{aligned} \tag{143}$$

for $\lambda > 1$. Thus Lebesgue's dominated convergence principle, ([9], p. 262), shows that we have

$$\begin{aligned}
&F_{(\alpha,\beta)}(x_1,x_2)\\
&= \int_{-1}^{1}\frac{u^{\alpha}}{(1+u^2)^{2\beta}}\, du\int_{-\infty}^{\infty}|\omega|\,\hat{\rho}[\omega^2(1+u^2)]\, e^{i(x_2+ux_1)\omega}\, d\omega\\
&+ \int_{-1}^{1}\frac{u^{4\beta-\alpha}}{(1+u^2)^{2\beta}}\, du\int_{-\infty}^{\infty}|\omega|\,\hat{\rho}[\omega^2(1+u^2)]\, e^{i(x_1+ux_2)\omega}\, d\omega\\
&= 2\int_{-\infty}^{\infty}\frac{u^{\alpha}}{(1+u^2)^{2\beta}}\, du\; \mathrm{Re}\left[\int_0^{\infty}\omega\,\hat{\rho}[\omega^2(1+u^2)]\, e^{i(x_1+x_2u)\omega}\, d\omega\right]
\end{aligned} \tag{144}$$

in which Re[·] means "real part." The last step follows by applying substitutions in the last double integral such that the integration parameters become $1/u$ and ωu in place of u and ω respectively. We write the result on the form

$$\begin{aligned}
F_{(\alpha,\beta)}(x_1,x_2) &= 2\int_{-\infty}^{\infty}\frac{u^{\alpha}}{(1+u^2)^{2\beta+1}}\,\psi(u,x_1,x_2)\, du\\
&= 2\int_{-1}^{1}\frac{u^{\alpha}\psi(u,x_1,x_2) + u^{4\beta-\alpha}\psi(u,x_2,x_1)}{(1+u^2)^{2\beta+1}}\, du
\end{aligned} \tag{145}$$

in which

$$\psi(u, x_1, x_2) = \mathrm{Re}\left[\int_0^\infty v\,\hat{\rho}(v^2)\exp\left(i\frac{x_2 + x_1 u}{\sqrt{1+u^2}}v\right)\mathrm{d}v\right]. \tag{146}$$

Since, eqn (63),

$$\rho(x_1^2 + x_2^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\rho}(\omega_1^2 + \omega_2^2)\,e^{i(x_1\omega_1 + x_2\omega_2)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \tag{147}$$

we have by reduction in polar coordinates that

$$\rho(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\rho}(\omega_1^2 + \omega_2^2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 = 2\pi\int_0^\infty v\,\hat{\rho}(v^2)\,\mathrm{d}v < \infty. \tag{148}$$

By comparison with eqn (146) this shows that

$$\psi(u, 0, 0) = \frac{\rho(0)}{2\pi} \tag{149}$$

giving

$$F_{(\alpha,\beta)}(0, 0) = \frac{\rho(0)}{\pi}\int_{-\infty}^{\infty}\frac{u^\alpha}{(1+u^2)^{2\beta+1}}\,\mathrm{d}u. \tag{150}$$

For $\rho(0) = 1$, the relevant values of $F_{(\alpha,\beta)}(0, 0)$ are given in the following table:

α	0	2	4	6	8	odd
$128F_{(\alpha,1)}$	48	16	48			0
$128F_{(\alpha,2)}$	35	5	3	5	35	0

Furthermore it follows from eqns (146) and (149) that

$$|\psi(u, x_1, x_2)| \le \rho(0)/2\pi. \tag{151}$$

It is worth noting the special case $\alpha = \beta = 0$. The inverse transform of $\hat{F}_{(0,0)}(\omega_1, \omega_2) = \hat{\rho}(\omega_1^2 + \omega_2^2)$ is $\rho(x_1^2 + x_2^2)$. Thus we have the representation, eqn (145),

$$\rho(x^2) = 2\int_{-\infty}^{\infty}\frac{\psi(u, ax, bx)}{1+u^2}\,\mathrm{d}u \tag{152}$$

valid for all a, b such that $a^2 + b^2 = 1$.

We will make use of the so-called Riemann–Lebesgue theorem of the theory of Fourier series and integrals ([10], p. 11). It states that

$$\int_0^\infty f(x)\,e^{i\lambda x}\,\mathrm{d}x \to 0 \tag{153}$$

for $\lambda \to \infty$ for any real function $f(x)$ for which $\int_0^\infty f(x)\,\mathrm{d}x < \infty$. By this theorem it follows from the convergence of the integral of eqn (148) that

$$\psi(u, x_1, x_2) \to 0 \quad \text{for} \quad x_1^2 + x_2^2 \to \infty. \tag{154}$$

Since $|\psi| \le \rho(0)/2\pi$, it follows by applying the dominated convergence principle on the integral of eqn (145) that

$$F_{(\alpha,\beta)}(x_1, x_2) \to 0 \quad \text{for} \quad x_1^2 + x_2^2 \to \infty. \tag{155}$$

APPENDIX 2

Some asymptotic results

We will next study the function

$$g(s) = \int_{-\infty}^{\infty} R(u)(au + b)^2\,\psi(u, as, bs)\,\mathrm{d}u \tag{156}$$

where $R(u)$ is a bounded rational function of u such that the integral is convergent for all $s \in R$, and a, b are given constants for which $a^2 + b^2 = 1$. In particular we are interested in the asymptotic behavior of

$$\frac{1}{L^2}\int_0^L (L - s)\,g(s)\,\mathrm{d}s \tag{157}$$

as $L \to \infty$. Since $|\psi| \le \rho(0)/2\pi < \infty$ we can commute the order of integration in the following integral

$$\int_0^L (L-s)\, g(s)\, ds = \int_{-\infty}^{\infty} R(u)(au+b)^2 \left[\int_0^L (L-s)\psi(u, as, bs)\, ds \right] du \tag{158}$$

and due to eqn (148) also in

$$\int_0^L (L-s)\, \psi(u, as, bs)\, ds = \int_0^{\infty} v\, \hat{\rho}(v^2) \left[\int_0^L (L-s) \cos(kvs)\, ds \right] dv \tag{159}$$

where

$$k = \frac{au+b}{\sqrt{1+u^2}}. \tag{160}$$

Integration by parts yields

$$\int_0^L (L-s)\cos(kvs)\, ds = \frac{1-\cos(kLv)}{(kv)^2} = \frac{2}{(kv)^2}\sin^2\left(\frac{kLv}{2}\right) \tag{161}$$

such that

$$\frac{1}{L^2}\int_0^L (L-s)\, \psi(u, as, bs)\, ds = \frac{1}{2}\int_0^{\infty} v\, \hat{\rho}(v^2) \left(\frac{\sin \frac{kLv}{2}}{\frac{kLv}{2}} \right)^2 dv \ge 0 \tag{162}$$

since $\hat{\rho}(v^2)$ is nonnegative everywhere. We will restrict the possible convariance functions to be considered herein to those for which $\rho(x^2) \ge 0$ for all x. This has the consequence that

$$\hat{\rho}(\omega_1^2 + \omega_2^2) \le \hat{\rho}(0) = \left(\frac{1}{2\pi}\right)^2 \tag{163}$$

since

$$\hat{\rho}(\omega_1^2 + \omega_2^2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x_1^2 + x_2^2)\, e^{-i(x_1\omega_1 + x_2\omega_2)}\, dx_1\, dx_2 \tag{164}$$

and since ρ is normalized such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x_1^2 + x_2^2)\, dx_1\, dx_2 = 1. \tag{165}$$

Consistent with this we assume that

$$\hat{\rho}(v^2) \le \left(\frac{1}{2\pi}\right)^2 \min\{1, v^{-2-\epsilon}\} \tag{166}$$

i.e. a condition with which the assumption of eqn (142) is consistent. By an obvious evaluation of the trigonometric factor in the integrand on the right side of eqn (162) we first get

$$0 \le \frac{1}{L^2}\int_0^L (L-s)\, \psi(u, as, bs)\, ds \le \frac{1}{2}\int_0^{\infty} v\, \hat{\rho}(v^2) \min\left\{1, \frac{4}{(kLv)^2}\right\} dv$$
$$= \frac{1}{2}\left[\int_0^{\delta} v\, \hat{\rho}(v^2)\, dv + \frac{4}{(kL)^2}\int_{\delta}^{\infty} \frac{1}{v}\hat{\rho}(v^2)\, dv \right] \tag{167}$$

in which $\delta = 2/|k|L$. Depending on whether $\delta \le 1$ or $\delta > 1$ we next get by using eqn (166)

$$\frac{1}{L^2}\int_0^L (L-s)\, \psi(u, as, bs)\, ds$$
$$\le \frac{1}{8\pi^2} \begin{cases} \int_0^{\delta} v\, dv + \delta^2 \int_{\delta}^{1} \frac{dv}{v} + \delta^2 \int_1^{\infty} \frac{dv}{v^{3+\epsilon}} \\ \int_0^{\delta} v\, dv + \delta^2 \int_{\delta}^{\infty} \frac{dv}{v^{3+\epsilon}} \end{cases}$$
$$\le \frac{\delta^2}{8\pi^2} \begin{cases} 1 + \log\frac{1}{\delta} & \text{for } \delta \le 1 \\ 1 & \text{for } \delta > 1 \end{cases}$$
$$= \frac{1}{2\pi^2 (kL)^2} \max\left\{1 + \log\frac{|k|L}{2}, 1\right\}. \tag{168}$$

It follows from eqns (158), (160), (162) and (168) that

$$\left| \frac{1}{L^2} \int_0^L (L - s)\, g(s)\, \mathrm{d}s \right|$$

$$\leq \frac{1}{2\pi^2 L^2} \int_{-\infty}^{\infty} | R(u) | (1 + u^2) \max \left\{ 1 + \log \left(\frac{| au + b |}{2\sqrt{1 + u^2}} \right) + \log L,\ 1 \right\} \mathrm{d}u$$

$$\leq \frac{1}{2\pi^2 L^2} \left[\int_{-\infty}^{\infty} | R(u) | (1 + u^2) \max \left\{ 1 + \log \left(\frac{| au + b |}{2\sqrt{1 + u^2}} \right),\ 1 \right\} \mathrm{d}u \right.$$

$$\left. + \int_{-\infty}^{\infty} | R(u) | (1 + u^2)\, \mathrm{d}u \log L \right]_{(\log L > 0)}$$

$$\propto \frac{1}{2\pi^2} \left(\int_{-\infty}^{\infty} | R(u) | (1 + u^2)\, \mathrm{d}u \right) \frac{\log L}{L^2} \tag{169}$$

where the last expression is valid asymptotically for large L. This result shows that

$$\frac{1}{\log L} \int_0^L (L - s)\, g(s)\, \mathrm{d}s \tag{170}$$

is bounded for $\log L > 1$, say. We will next show that it has a finite limit for $L \to \infty$. We need the following

LEMMA. Let f be a nonnegative real function defined on $[0, \infty[$. If f is continuous at zero and

$$\int_\delta^\infty \frac{f(x)}{x}\, \mathrm{d}x < \infty \tag{171}$$

for any positive δ, and if c is a positive constant, then

$$\lim_{\lambda \to \infty} \int_0^\infty f(x) \frac{\sin^2 \lambda x}{x \log (\lambda c)}\, \mathrm{d}x = \tfrac{1}{2} f(0). \tag{172}$$

Proof. For an $\epsilon > 0$ choose $\delta > 0$ such that $f(0) - \epsilon \leq f(x) \leq f(0) + \epsilon$ for all $x \in [0, \delta]$. For $\lambda > \pi/\delta$ we then have

$$\int_0^{\pi/\lambda} f(x) \frac{\sin^2 \lambda x}{x \log (\lambda c)}\, \mathrm{d}x \leq \int_0^{\pi/\lambda} f(x) \frac{(\lambda x)^2}{x \log (\lambda c)}\, \mathrm{d}x$$

$$= \frac{\displaystyle\int_0^{\pi/\lambda} x f(x)\, \mathrm{d}x}{\dfrac{\log (\lambda c)}{\lambda^2}} \leq \frac{(f(0) + \epsilon) \dfrac{\pi^2}{2}}{\log (\lambda c)} \to 0 \tag{173}$$

for $\lambda \to \infty$. Since

$$\int_{\pi/\lambda}^{\delta} \frac{f(x)}{x} \frac{\cos^2 \lambda x + \sin^2 \lambda x}{\log (\lambda c)}\, \mathrm{d}x \begin{Bmatrix} \leq (f(0) + \epsilon) \\ \geq (f(0) - \epsilon) \end{Bmatrix} \times \frac{\log \left(\dfrac{\delta \lambda}{\pi} \right)}{\log (\lambda c)} \to \begin{Bmatrix} f(0) + \epsilon \\ f(0) - \epsilon \end{Bmatrix} \tag{174}$$

and

$$\int_\delta^\infty f(x) \frac{\sin^2 \lambda x}{x \log (\lambda c)}\, \mathrm{d}x \leq \frac{1}{\log (\lambda c)} \int_\delta^\infty \frac{f(x)}{x}\, \mathrm{d}x \to 0 \tag{175}$$

we only need to show that

$$\int_{\pi/\lambda}^{\delta} \frac{f(x)}{x} \frac{\cos^2 \lambda x - \sin^2 \lambda x}{\log (\lambda c)}\, \mathrm{d}x \to 0 \tag{176}$$

for $\lambda \to \infty$ in order to complete the proof of eqn (172). For $\lambda > 3\pi/2\delta$ we get

$$\int_{\pi/\lambda}^{\delta} \frac{f(x)}{x} (\cos^2 \lambda x - \sin^2 \lambda x)\, \mathrm{d}x$$

$$= \int_{\pi/\lambda}^{\delta} \frac{f(x)}{x} \left(\sin^2 \lambda \left(x + \frac{\pi}{2\lambda} \right) - \sin^2 \lambda x \right) \mathrm{d}x$$

$$= - \int_{\pi/\lambda}^{3\pi/2\lambda} \frac{f(x)}{x} \sin^2 \lambda x\, \mathrm{d}x + \int_{3\pi/2\lambda}^{\delta} \left(\frac{f\left(x - \dfrac{\pi}{2\lambda} \right)}{x - \dfrac{\pi}{2\lambda}} - \frac{f(x)}{x} \right) \sin^2 \lambda x\, \mathrm{d}x$$

$$\begin{cases} \le \int_{3\pi/2\lambda}^{\delta} \left(\dfrac{f(0)+\epsilon}{x-\dfrac{\pi}{2\lambda}} - \dfrac{f(0)-\epsilon}{x} \right) dx \le f(0)\log\dfrac{3}{2} + \epsilon\left[\log\dfrac{2}{3}\left(\dfrac{\delta\lambda}{\pi}\right)^2\right] \\ \ge -\int_{\pi/\lambda}^{\delta} \dfrac{f(0)+\epsilon}{x}\, dx = -(f(0)+\epsilon)\log\left(\dfrac{\delta\lambda}{\pi}\right). \end{cases} \tag{177}$$

After division by log (λc) the upper bound approaches 2ϵ while the lower bound approaches zero for $\lambda \to \infty$. Since ϵ is arbitrary, the lemma follows.

After multiplication of eqn (162) by $L^2/\log L$ we may apply the lemma to get

$$\frac{1}{\log L}\int_0^L (L-s)\,\psi(u, as, bs)\, ds = \frac{2}{k^2}\int_0^c \hat\rho(v^2)\,\frac{\sin^2\dfrac{kLv}{2}}{v\log L}\, dv \to \frac{\hat\rho(0)}{k^2} = \frac{1}{(2\pi)^2 k^2} \tag{178}$$

for $L \to \infty$. By using the inequality of eqn (168) it is seen that an absolute bound to the integrand of eqn (158) after division by log L is

$$\frac{1}{(2\pi)^2}\,|R(u)|\,(1+u^2)\max\left\{2+\log\left(\frac{|au+b|}{2\sqrt{1+u^2}}\right), 1\right\} \tag{179}$$

Since this function of u is integrable from $-\infty$ to ∞, the dominated convergence principle finally shows that

$$\boxed{\lim_{L\to\infty}\frac{1}{\log L}\int_0^L (L-s)\int_{-\infty}^{\infty} R(u)\,(au+b)^2\,\psi(u, as, bs)\, du\, ds = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} R(u)\,(1+u^2)\, du.} \tag{180}$$

By removal of the factor $(au+b)^2$ in the integrand of eqn (156) we get the function

$$h(s) = \int_{-\infty}^{\infty} R(u)\,\psi(u, as, bs)\, du \tag{181}$$

for which the integral of eqn (156) with h in place of g behaves asymptotically quite differently than stated by eqn (180), except for $a = 0$, of course. In eqn (181), $R(u)$ is a bounded rational function of order of magnitude $O(1/u^2)$ for $|u| \to \infty$.

The calculation up to eq (162) is the same giving

$$\begin{aligned} &\frac{1}{L}\int_0^L (L-s)\,h(s)\, ds \\ &= \frac{L}{2}\int_{-\infty}^{\infty} R(u)\left[\int_0^{\infty} v\,\hat\rho(v^2)\left(\frac{\sin\dfrac{kLv}{2}}{\dfrac{kLv}{2}}\right)^2 dv\right] du \\ &= \frac{1}{2}\int_0^{\infty}\hat\rho(v^2)\left[L\int_{-\infty}^{\infty} v\,R(u)\left(\frac{\sin\dfrac{kLv}{2}}{\dfrac{kLv}{2}}\right)^2 du\right] dv \end{aligned} \tag{182}$$

where the interchange of order of integration is admissible since the function

$$G(u, v) = R(u)\left(\frac{\sin\dfrac{kLv}{2}}{\dfrac{kLv}{2}}\right)^2 \tag{183}$$

is absolutely bounded by a constant and eqn (148) is valid, and further since

$$\int_{-\infty}^{\infty} |G(u, v)|\, du \le \int_{-\infty}^{\infty} |R(u)|\, du < \infty. \tag{184}$$

The reader is referred to ([4], pp. 66–70), e.g. for the proof of the sufficiency of these conditions.

The inner integral is by the substitution $y = (au+b)/\sqrt{1+u^2}$ changed into

$$\begin{aligned} &Lv\int_{-\infty}^{\infty} R(u)\left(\frac{\sin\dfrac{(au+b)Lv}{2\sqrt{1+u^2}}}{\dfrac{(au+b)Lv}{2\sqrt{1+u^2}}}\right)^2 du \\ &= \lim_{\delta\downarrow 0}\left[Lv\int_{y=-a}^{1-\delta} R(u_1)\,\frac{(1+u_1^2)^{3/2}}{a-bu_1}\left(\frac{\sin\frac{1}{2}yLv}{\frac{1}{2}yLv}\right)^2 dy\right] \\ &+ \lim_{\delta\downarrow 0}\left[Lv\int_{y=1-\delta}^{a} R(u_2)\,\frac{(1+u_2^2)^{3/2}}{a-bu_2}\left(\frac{\sin\frac{1}{2}yLv}{\frac{1}{2}yLv}\right)^2 dy\right] \end{aligned} \tag{185}$$

in which u_1, u_2 are the functions of y defined by solving the substitution with respect to y. The first function corresponds to $-\infty < u < a/b$ while the second function corresponds to $a/b < u < \infty$.

Since

$$\int_{y=-a}^{1-\delta} |R(u_1)| \frac{(1+u_1^2)^{3/2}}{|a-bu_1|} \mathrm{d}y \le \int_{-\infty}^{a/b} |R(u)| \mathrm{d}u \tag{186}$$

$$\int_{y=a}^{1-\delta} |R(u_2)| \frac{(1+u_2^2)^{3/2}}{|a-bu_2|} \mathrm{d}y \le \int_{a/b}^{\infty} |R(u)| \mathrm{d}u \tag{187}$$

and

$$\left.\begin{matrix}\int_{-a}^{1-\delta} \\ \int_{a}^{1-\delta}\end{matrix}\right\} Lv \left(\frac{\sin \frac{1}{2}yLv}{\frac{1}{2}yLv}\right)^2 \mathrm{d}y \le 2\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = 2\pi \tag{188}$$

we see that the integral of eqn (185) has the absolute bound

$$4\pi \int_{-\infty}^{\infty} |R(u)| \mathrm{d}u. \tag{189}$$

Since $\int_0^\infty \hat{\rho}(v^2)\,\mathrm{d}v < \infty$, eqn (166), it follows from the dominated convergence principle that we may pass to the limit $L \to \infty$ behind the first integral of eqn (182). We may now write the left side of eqn (185) as the sum of the integral

$$\frac{4}{Lv} \int_{u_1(1-\delta)}^{u_2(1-\delta)} R(u) \frac{1+u^2}{(au+b)^2} \sin^2\left(\frac{(au+b)Lv}{2\sqrt{1+u^2}}\right) \mathrm{d}u \tag{190}$$

and the two integrals in the brackets on the right side of eqn (185). The positive number δ is selected so small that $-b/a$ is not in the closed interval from $u_1(1-\delta)$ to $u_2(1-\delta)$. It is obvious that the integral of eqn (190) approaches zero for $L \to \infty$. The same applies to the integral in the second bracket on the right side of eqn (185) since the interval of integration does not contain $y = 0$. Of the same reason the limit of the integral of the first integral is the same as

$$\lim_{L\to\infty} \left[Lv \int_{y=-\gamma}^{\gamma} R(u_1) \frac{(1+u_1^2)^{3/2}}{a-bu_1} \left(\frac{\sin \frac{1}{2}yLv}{\frac{1}{2}yLv}\right)^2 \mathrm{d}y \right] \tag{191}$$

for any arbitrary sufficiently small positive value of γ. Since the integrand is a continuous function of y for $y = 0$, it follows that the limit is

$$2R(u_1(0)) \frac{(1+u_1(0)^2)^{3/2}}{a-bu_1(0)} \lim_{L\to\infty} \int_{-\gamma Lv/2}^{\gamma Lv/2} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = 2\pi R\left(-\frac{b}{a}\right)\frac{1}{a^2} \tag{192}$$

except for $v = 0$, in which case the limit is zero. Finally, applying this in eqn (182) we get the result

$$\boxed{\lim_{L\to\infty} \frac{1}{L} \int_0^L (L-s) \int_{-\infty}^{\infty} R(u)\,\psi(u, as, bs)\,\mathrm{d}u\,\mathrm{d}s = \pi R\left(-\frac{b}{a}\right)\frac{1}{a^2}\int_0^\infty \hat{\rho}(v^2)\,\mathrm{d}v} \tag{193}$$

valid for $a > 0$. For $a = 0$ it follows by use of eqn (180) that the limit is zero since $(\log L)/L \to 0$ for $L \to \infty$.

By application of eqn (193) on eqn (152) we get

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L (L-x)\,\rho(x^2)\,\mathrm{d}x = 2\pi \int_0^\infty \hat{\rho}(v^2)\,\mathrm{d}v. \tag{194}$$

Since $\int_0^\infty x\,\rho(x^2)\,\mathrm{d}x = 1/2\pi$, eqn (165), this gives the formula

$$\int_0^\infty \rho(x^2)\,\mathrm{d}x = 2\pi \int_0^\infty \hat{\rho}(v^2)\,\mathrm{d}v. \tag{195}$$

APPENDIX 3

Two specific examples

The Gaussian type covariance function, eqn (66),

$$\rho(x^2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}x^2\right] \tag{196}$$

has the transform

$$\hat{\rho}(v^2) = \left(\frac{1}{2\pi}\right)^2 \exp\left[-\frac{1}{2}v^2\right] \tag{197}$$

giving, eqn (146),

$$\psi(u, x_1, x_2) = \left(\frac{1}{2\pi}\right)^2 \operatorname{Re}\left[\int_0^\infty x\, e^{-x^2/2 - iyx}\, dx\right] \tag{198}$$

with

$$y = -\frac{x_2 + x_1 u}{\sqrt{1 + u^2}}. \tag{199}$$

The integral of eqn (198) is

$$e^{-y^2/2} \int_0^\infty x\, e^{-(x+iy)^2/2}\, dx. \tag{200}$$

Contour integration gives

$$\begin{aligned}\int_0^\infty x\, e^{-(x+iy)^2/2}\, dx &= \int_{iy}^{\infty+iy} z\, e^{-z^2/2}\, dz - iy \int_{iy}^{\infty+iy} e^{-z^2/2}\, dz \\ &= [-e^{-z^2/2}]_{iy}^{\infty+iy} - iy\left[-\int_0^y e^{t^2/2}\, dt + \int_0^\infty e^{-x^2/2}\, dx\right] \\ &= e^{y^2/2} - y\int_0^y e^{t^2/2}\, dt - iy\sqrt{\frac{\pi}{2}}.\end{aligned} \tag{201}$$

Thus the real part of the integral is

$$1 - y^2 \int_0^1 e^{-y^2(1-v^2)/2}\, dv \tag{202}$$

such that eqn (198) becomes

$$(2\pi)^2 \psi(u, x_1, x_2) = 1 - \frac{(x_2 + x_1 u)^2}{1 + u^2} \int_0^1 \exp\left[-\tfrac{1}{2}(x_2 + x_1 u)^2 \frac{1 - v^2}{1 + u^2}\right] dv \tag{203}$$

It seems not to be easy to reduce this formula further.

Another example of a covariance function is the Cauchy type, eqn (67),

$$\rho(x^2) = \frac{1}{2\pi}[1 + x^2]^{-3/2} \tag{204}$$

with the transform

$$\hat{\rho}(v^2) = \left(\frac{1}{2\pi}\right)^2 \exp[-|v|]. \tag{205}$$

In this case eqn (146) becomes

$$\psi(u, x_1, x_2) = \left(\frac{1}{2\pi}\right)^2 \operatorname{Re}\left[\int_0^\infty x\, e^{-(1+iy)x}\, dx\right] \tag{206}$$

in which the integral is

$$\frac{1}{(1 + iy)^2}\int_0^\infty z\, e^{-z}\, dz = \left(\frac{1 - iy}{1 + y^2}\right)^2 = \frac{1 - y^2}{(1 + y^2)^2} - i\frac{2y}{(1 + y^2)^2} \tag{207}$$

such that eqn (206) becomes

$$(2\pi)^2 \psi(u, x_1, x_2) = \frac{1 - \dfrac{(x_2 + x_1 u)^2}{1 + u^2}}{\left(1 + \dfrac{(x_2 + x_1 u)^2}{1 + u^2}\right)^2}. \tag{208}$$